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# Solution of the $H_3^+$ model on a disc

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Abstract: We determine all the correlators of the $H_3^+$ model on a disc with $AdS_2$-brane boundary conditions in terms of correlators of Liouville theory on a disc with FZZT-brane boundary conditions. We argue that the Cardy-Lewellen constraints are weaker in the $H_3^+$ model than in rational conformal field theories due to extra singularities of the correlators, but strong enough to uniquely determine the bulk two-point function on a disc. We confirm our results by detailed analyses of the bulk-boundary two-point function and of the boundary two-point function. In particular we find that, although the target space symmetry preserved by $AdS_2$-branes is the group $\mathrm{SL}(2,\mathbb{R})$, the open string states between two distinct parallel $AdS_2$-branes belong to representations of the universal covering group.

Keywords: D-branes, Conformal Field Models in String Theory, Conformal and W Symmetry.





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## Contents



## 1. Introduction and summary

String theory in $AdS_3$ plays an important rôle in building string theory models of black holes and cosmology, and in the AdS/CFT correspondence. The $AdS_3$ space-time is interesting because it is Lorentzian, non-compact, and curved; the theory is nevertheless expected to be tractable thanks to the $\widehat{s\ell_2}$ affine symmetry. However, the Lorentzian feature is still a major technical hurdle. It can be avoided by Wick rotation, which makes space-time Euclidean while still non-compact and curved, and relates the theory to the $H_3^+$ model,



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i.e. string theory in the Euclidean $AdS_3$, which still has the $\widehat{s\ell_2}$ symmetry. Solving the $H_3^+$ model is therefore a crucial technical step in the study of string theory in $AdS_3$. By solving the model we mean determining its spectrum and arbitrary correlators on arbitrary Riemann surfaces. As was shown in [1, 2], the partition function and the correlators can be defined, and in a few simple cases computed, within a path integral formulation. The $H_3^+$ model has also been studied using the conformal bootstrap formalism [3], which exploits consistency constraints (like crossing symmetry) on these correlators. These consistency constraints were shown to be sufficient for fully determining the correlators of the $H_3^+$ model on a sphere, and explicitly computing the three-point function [4, 5].

The usefulness of the conformal bootstrap formalism for the $H_3^+$ model on a sphere was not obvious from the start, since the formalism was developed for rational conformal field theories whereas the $H_3^+$ model is a non-rational CFT with a continuous spectrum. However, this formalism also brought about significant progress in the case of Liouville theory on the sphere and on the disc. Liouville theory is a simpler non-rational CFT which can be considered as solved in the sense of the bootstrap formalism, because some elementary correlators were explicitly computed, in terms of which all the other correlators can in principle be deduced.

Encouraged by these examples, one might expect the $H_3^+$ model on a disc to be solvable by means of the conformal bootstrap formalism. As was first noticed in [6], there is however a problem due to the presence of singularities in some correlators. These singularities weaken the Cardy-Lewellen constraints [7, 8], i.e. the conformal bootstrap equations on the disc. Such singularities are a consequence of the $\widehat{s\ell_2}$ symmetry of the model and are therefore also present in the $H_3^+$ model on the sphere, where they can however be circumvented by analytic continuation.

We will show how the $H_3^+$ model on the disc can be solved in spite of these singularities. The main tool which enables us to analyze the singularities and solve the model is the $H_3^+$-Liouville relation, which was first established in the case of the sphere [9]. In particular, our main result is a formula (3.18) for arbitrary correlators of the $H_3^+$ model at level $k > 2$ on the disc, in terms of correlators of Liouville theory at parameter $b^2 = (k-2)^{-1}$ on the disc. Schematically, (3.18) reads:

$$\left\langle \prod_{a=1}^{n} \Phi^{j_a} \prod_{b=1}^{m} {}_{r_{b-1,b}}\Psi^{\ell_b}{}_{r_{b,b+1}} \right\rangle \propto \left\langle \prod_{a=1}^{n} V_{\alpha_a} \prod_{b=1}^{m} {}_{s_{b-1,b}}\left(B_{\beta_b}\right)_{s_{b,b+1}} \prod_{a'=1}^{n'} V_{-\frac{1}{2b}} \prod_{b'=1}^{m'} B_{-\frac{1}{2b}} \right\rangle , \tag{1.1}$$

where $\Phi^{j_a}, \Psi^{\ell_b}$ are $H_3^+$ bulk and boundary fields with spins $j_a, \ell_b$ respectively, $V_{\alpha_a}, B_{\beta_b}$ corresponding Liouville bulk and boundary fields with corresponding momenta $\alpha_a, \beta_b$ respectively, and $V_{-\frac{1}{2b}}, B_{-\frac{1}{2b}}$ are extra degenerate Liouville fields. The boundary conditions are maximally symmetric in both theories, they correspond to $AdS_2$ branes [10] in $H_3^+$ with parameters $r$ and FZZT branes [11, 12] in Liouville theory with parameters $s = \frac{r}{2\pi b} \pm \frac{i}{4b}$. We are able to prove the formula for all correlators which do not involve boundary condition changing operators, leaving the remaining cases as a strongly supported conjecture. We also reformulate our result suitably for its application to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model (3.32).

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Since all Liouville correlators on the disc are known in principle, our $H_3^+$-Liouville relation on the disc amounts to a solution of the $H_3^+$ model on the disc. For some correlators, the conformal blocks are simple enough that explicit expressions can be found. We will write such explicit expressions in the cases of the boundary two-point function (section 4) and the bulk-boundary two-point function (section 5). These special cases will allow us to perform some consistency checks: comparing the boundary two-point function with predictions of the classical $H_3^+$ model and with N=2 Liouville theory, and the bulk-boundary two-point function with a minisuperspace analysis.

Finding the boundary two-point function amounts to determining the spectrum of open strings stretched between two $AdS_2$ branes. In the case when these two branes are different, we encounter a surprise: our results are incompatible with the $\mathrm{SL}(2,\mathbb{R})$ symmetry which was previously assumed for this system, and show that the correct symmetry group is the universal covering group $\widetilde{SL}(2,\mathbb{R})$.

The section 2 and the appendices provide supporting material on the $H_3^+$ model, special functions, and Liouville theory.

## 2. The $H_3^+$ model: state of the art

Let us review known results about the $H_3^+$ model on Riemann surfaces without boundaries, or with boundaries defined by $AdS_2$ branes.

### 2.1 Bulk $H_3^+$ model

The space $H_3^+$ is a three-dimensional hyperboloid, or equivalently the space of $(2 \times 2)$ Hermite matrices $h$ with unit determinant and positive trace:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\,, \quad x_0 > 0\ ; \quad h = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}\ . \tag{2.1}$$

The $H_3^+$ model at level $k$ on a two-dimensional Riemann surface $\Sigma$ parametrized by $z$ can be defined by the WZW-like action [2] of a matrix field $h(z,\bar{z})$,

$$S^H[h] \;=\; \frac{k}{2\pi}\int_\Sigma d^2z \mathrm{Tr}[h^{-1}\partial h h^{-1}\bar{\partial}h] + \frac{k}{12\pi i}\int_{\partial^{-1}\Sigma} \mathrm{Tr}(h^{-1}dh)^3\ . \tag{2.2}$$

The $H_3^+$ model is therefore a sigma-model with the manifold $H_3^+$ as target space, and a non-trivial $B$-field. One often parametrizes $H_3^+$ by coordinates $\phi, \gamma, \bar{\gamma}$ as

$$h = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix}\begin{pmatrix} 1 & \bar{\gamma} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\phi & e^\phi\bar{\gamma} \\ e^\phi\gamma & e^\phi\gamma\bar{\gamma} + e^{-\phi} \end{pmatrix}\ . \tag{2.3}$$

In terms of these coordinates the action becomes

$$S^H \;=\; \frac{k}{\pi}\int d^2z\ \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\gamma\bar{\partial}\bar{\gamma}\right)\ . \tag{2.4}$$

The symmetry of the $H_3^+$ model includes the $\mathrm{SL}(2,\mathbb{C})$ isometry group of the $H_3^+$ manifold. The action of an $\mathrm{SL}(2,\mathbb{C})$ group element $g$ on $H_3^+$ is $g \cdot h \equiv ghg^\dagger$, so the element

$g = -\mathrm{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $H_3^+$. Thus the non-trivially acting isometry group is actually $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,3)$. The isometry group $\mathrm{SO}(1,3)$ also follows from the definition of $H_3^+$ as an hyperboloid.

In the "minisuperspace" limit [13], which involves sending the level $k$ to infinity, the spectrum of the model reduces to the space of functions on the $H_3^+$ manifold parametrized by $(\phi, \gamma, \bar{\gamma})$. This minisuperspace spectrum is generated by the following functions:

$$\Phi^j(x|h) = \frac{2j+1}{\pi} \left( [-x \ 1] h \begin{bmatrix} -\bar{x} \\ 1 \end{bmatrix} \right)^{2j} = \frac{2j+1}{\pi} \left( |\gamma - x|^2 e^{\phi} + e^{-\phi} \right)^{2j} , \tag{2.5}$$

where delta-function normalizability requires $j \in -\frac{1}{2} + i\mathbb{R}$. This number $j$ is the spin of an $\mathrm{SL}(2,\mathbb{C})$ representation; states belonging to the same representation are parametrized by the isospin variable $x \in \mathbb{C}$. The behaviour of $\Phi^j$ under an $\mathrm{SL}(2,\mathbb{C})$ transformation $g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ is

$$\Phi^j(x|g \cdot h) = |\gamma x - \delta|^{4j} \Phi^j(g \cdot x|h) \, , \ g \cdot x = \frac{\alpha x - \beta}{\gamma x - \delta} \, . \tag{2.6}$$

The spectrum of the quantum $H_3^+$ model [2, 4] can formally be built from the minisuperspace spectrum by acting with oscillators encoding the worldsheet $z$-dependence, which amounts to extending the representations of the group $\mathrm{SL}(2,\mathbb{C})$ into representations of the corresponding loop group. The set of physical representations itself does not change; unless specified otherwise our integrals on the spin $j$ will be over this set $j \in -\frac{1}{2} + i\mathbb{R}$. The conformal weight of the primary field $\Phi^j(x|z)$ built from the classical field $\Phi^j(x|h)$ is (using the notation $b^2 = (k-2)^{-1}$)

$$\Delta_j = -b^2 j(j+1) = -\frac{j(j+1)}{k-2} \, . \tag{2.7}$$

The symmetry algebra of the $H_3^+$ model is (after complexification) the affine Lie algebra $\widehat{s\ell_2} \times \widehat{s\ell_2}$ generated by the modes of the currents $J = k\partial h h^{-1}$, $\bar{J} = kh^{-1}\bar{\partial} h$. This symmetry results in the correlators obeying the Knizhnik-Zamolodchikov equations, which we will recall and use in section 3. For now, let us write the consequences of the global symmetry group $\mathrm{SL}(2,\mathbb{C})$:

$$\left\langle \prod_{a=1}^{n} g \cdot \Phi^{j_a}(x_a|z_a) \right\rangle = \left\langle \prod_{a=1}^{n} \Phi^{j_a}(x_a|z_a) \right\rangle , \tag{2.8}$$

where the $\mathrm{SL}(2,\mathbb{C})$ transformation of the quantum field is defined by

$$g \cdot \Phi^j(x|z) \equiv |\gamma x - \delta|^{4j} \Phi^j(g \cdot x|z) \, . \tag{2.9}$$

Due to this simple transformation law, the isospin variable $x$ is very convenient for the study of the $\mathrm{SL}(2,\mathbb{C})$ symmetry. But for the purpose of writing $H_3^+$ correlators in terms of



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Liouville correlators it is more convenient to use the Fourier-transformed $\mu$-basis [9]

$$\Phi^j(\mu|z) = \frac{1}{\pi}|\mu|^{2j+2}\int_{\mathbb{C}} d^2x\ e^{\mu x-\bar{\mu}\bar{x}}\Phi^j(x|z)\ . \tag{2.10}$$

And for the purpose of comparing the $H_3^+$ model with N=2 supersymmetric Liouville theory, we will need the $m$-basis

$$\Phi^j_{m\bar{m}}(z) = \int \frac{d^2x}{|x|^2} x^{-j+m}\bar{x}^{-j+\bar{m}}\Phi^j(x|z) = N^j_{m\bar{m}}\int \frac{d^2\mu}{|\mu|^2}\mu^{-m}\bar{\mu}^{-\bar{m}}\Phi^j(\mu|z)\,, \tag{2.11}$$

where the physical values of $m, \bar{m}$ and the normalization $N^j_{m\bar{m}}$ are

$$m = \frac{n+ip}{2},\ \bar{m} = \frac{-n+ip}{2},\ (n,p)\in\mathbb{Z}\times\mathbb{R},\ N^j_{m\bar{m}} = \frac{\Gamma(-j+m)}{\Gamma(j+1-\bar{m})}\ . \tag{2.12}$$

Some basic correlators of the $H_3^+$ model on a sphere can be written explicitly. The bulk two-point function is

$$\begin{aligned}\left\langle\Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\right\rangle = |z_2-z_1|^{-4\Delta_{j_1}}|\mu_1|^2\delta^{(2)}(\mu_2+\mu_1)&\\ \times\left(\delta(j_2+j_1+1)+R^H(j_1)\delta(j_2-j_1)\right),&\end{aligned} \tag{2.13}$$

where we introduce the bulk reflection coefficient $R^H(j)$ such that

$$\Phi^j(\mu|z) = R^H(j)\Phi^{-j-1}(\mu|z)\,,\ R^H(j) = b^{-2}\left(\frac{1}{\pi}b^2\gamma(b^2)\right)^{-(2j+1)}\frac{\gamma(+2j+1)}{\gamma(-b^2(2j+1))}\ . \tag{2.14}$$

The bulk three-point function [4] is here written in the $\mu$-basis in a manifestly reflection-covariant way [14]:

$$\begin{aligned}\left\langle\prod_{a=1}^{3}\Phi^{j_a}(\mu_a|z_a)\right\rangle &= \frac{\delta^{(2)}(\sum\mu_a)}{|z_{12}|^{2\Delta_{12}^3}|z_{13}|^{2\Delta_{13}^2}|z_{23}|^{2\Delta_{23}^1}}D^H\begin{bmatrix} j_1 & j_2 & j_3\\ \mu_1 & \mu_2 & \mu_3\end{bmatrix}C^H(j_1,j_2,j_3),\\ D^H &= \frac{|\mu_1|^{2j_1+2}}{|\mu_2|^{2j_1}}\sum_{\eta=\pm}\gamma^{j_1,j_2}_{j_3^\eta}\left|\frac{\mu_2}{\mu_3}\right|^{2j_3^\eta}\ {}_2\mathcal{F}_1\big(j_1-j_2-j_3^\eta, j_1+j_2-j_3^\eta+1,-2j_3^\eta;-\frac{\mu_3}{\mu_2}\big),\\ C^H &= -\frac{1}{2\pi^2 b}\left[\frac{\gamma(b^2)b^{2-2b^2}}{\pi}\right]^{-2-\Sigma j_i}\frac{\Upsilon_b'(0)}{\Upsilon_b(-b(j_{123}+1))\Gamma(-j_{123}-1)}\\ &\quad\times\frac{\Upsilon_b(-b(2j_1+1))\Upsilon_b(-b(2j_2+1))\Upsilon_b(-b(2j_3+1))}{\Upsilon_b(-bj_{12}^3)\Gamma(-j_{12}^3)\ \Upsilon_b(-bj_{13}^2)\Gamma(-j_{13}^2)\ \Upsilon_b(-bj_{23}^1)\Gamma(-j_{23}^1)}\ .\end{aligned} \tag{2.15}$$

$$\text{Notations:}\begin{cases}\Delta_{12}^3 = \Delta_{j_1}+\Delta_{j_2}-\Delta_{j_3},\ \ j_{12}^3 = j_1+j_2-j_3,\ \ j_{123} = j_1+j_2+j_3,\\ j^+ = j,\ \ j^- = -j-1,\ \ {}_2\mathcal{F}_1(a,b,c;z) = F(a,b,c;z)F(a,b,c;\bar{z}),\\ \gamma^{j_1,j_2}_{j_3} = \Gamma(-j_{123}-1)\Gamma(-j_{23}^1)\Gamma(-j_{13}^2)\Gamma(j_{12}^3+1)\gamma(2j_3+1).\end{cases}$$

The special functions $\gamma$ and $\Upsilon_b$ are defined in the appendix. The reflection covariance of this expression follows from the reflection invariance of $D^H$, and the reflection behaviour $C^H(j_1, j_2, j_3) = R^H(j_3)C^H(j_1, j_2, -j_3 - 1)$.

The four-point function of the $H_3^+$ model has been shown to be crossing symmetric [5]. This means that it can be deduced from the three-point structure constant $C^H$ in two different ways:

$$\left\langle \prod_{a=1}^{4} \Phi^{j_a}(\mu_a|z_a) \right\rangle = \int dj_s\ C^H(j_1, j_2, j_s)\ C^H(-j_s - 1, j_3, j_4)\ \mathcal{G}^s_{j_s}(j_a|\mu_a|z_a) \tag{2.16}$$

$$= \int dj_t\ C^H(j_1, j_4, j_t)\ C^H(-j_t - 1, j_2, j_3)\ \mathcal{G}^t_{j_t}(j_a|\mu_a|z_a)\,, \tag{2.17}$$

where the $s$ and $t$-channel conformal blocks $\mathcal{G}^s_{j_s}(j_a|\mu_a|z_a)$ and $\mathcal{G}^t_{j_t}(j_a|\mu_a|z_a)$ are entirely determined by the affine $\widehat{s\ell_2}$ symmetry and thus in principle known before solving the model. This crossing symmetry relation should be viewed as a constraint on the three-point structure constant $C^H$. Exploiting very special cases of this constraint was enough to unambiguously determine $C^H$ [4]. That this unique solution turned out to satisfy the full crossing symmetry was an additional non-trivial check.

### 2.2 Euclidean $AdS_2$ branes

Euclidean $AdS_2$ branes preserve an $\mathrm{SL}(2,\mathbb{R})$ subgroup of the bulk symmetry group $\mathrm{SL}(2,\mathbb{C})$ [10]. The geometry of these D-branes is defined by the equation

$$\mathrm{Tr}\ \Omega h = 2 \sinh r\,, \tag{2.18}$$

for $r$ a real parameter, and $\Omega$ a Hermitian matrix which determines the relevant $\mathrm{SL}(2,\mathbb{R})$ subgroup as the set of $\mathrm{SL}(2,\mathbb{C})$ matrices such that $g^\dagger \Omega g = \Omega$. For definiteness we choose $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, in which case the $\mathrm{SL}(2,\mathbb{R})$ subgroup is the set of matrices

$$g = \begin{pmatrix} a & ic \\ -ib & d \end{pmatrix},\ ad - bc = 1,\ \ a, b, c, d \in \mathbb{R}\ . \tag{2.19}$$

In the minisuperspace limit, the spectrum of open strings on an $AdS_2$ brane reduces to the space of functions on the corresponding two-dimensional submanifold of $H_3^+$. The minisuperspace spectrum is generated by the functions:

$$\Psi^\ell(t|h) = \left( [\,it\ 1\,] h \begin{bmatrix} -it \\ 1 \end{bmatrix} \right)^\ell\,, \tag{2.20}$$

where the boundary spin $\ell$ belongs to $-\frac{1}{2} + i\mathbb{R}$, and the boundary isospin is $t \in \mathbb{R}$. (For more details see appendix A.2 of [10].) Under $\mathrm{SL}(2,\mathbb{R})$ transformations we have

$$\Psi^\ell(t|g \cdot h) = |ct - d|^{2\ell}\Psi^\ell(g \cdot t|h)\,,\ \ g \cdot t = \frac{at - b}{-ct + d}\ . \tag{2.21}$$

The spectrum of the quantum model is generated by corresponding boundary fields $\Psi^\ell(t|w)$ with $w$ a real coordinate on the worldsheet boundary, which transform as

$$g \cdot \Psi^\ell(t|w) \equiv |ct-d|^{2\ell}\Psi^\ell(g\cdot t|w)\ . \tag{2.22}$$

There also exist $\mathrm{SL}(2,\mathbb{R})$ representations whose fields would behave as $g\cdot\Psi^\ell(t|w) = |ct-d|^{2\ell}\mathrm{sgn}(-ct+d)\Psi^\ell(g\cdot t|w)$, but such fields do not appear in the minisuperspace spectrum of $AdS_2$ branes and we assume that they are absent from the exact spectrum as well. We will naturally assume that correlators involving boundary fields preserve the $\mathrm{SL}(2,\mathbb{R})$ symmetry:

$$\left\langle \prod_{a=1}^{n} g\cdot\Phi^{j_a}(x_a|z_a)\prod_{b=1}^{m} g\cdot\Psi^{\ell_b}(t_b|w_b)\right\rangle = \left\langle \prod_{a=1}^{n} \Phi^{j_a}(x_a|z_a)\prod_{b=1}^{m} \Psi^{\ell_b}(t_b|w_b)\right\rangle\ . \tag{2.23}$$

We will also be interested in boundary condition changing fields ${}_{r'}\Psi^\ell(t|w)_r$ describing open strings stretched between two $AdS_2$ branes with different parameters $r, r'$. We will see in section 4 that the symmetry properties of these fields are significantly more complicated. So in the present review section we focus on the already well-understood $r$-preserving fields.

The $t$-basis boundary fields we have considered so far are useful for the study of the $\mathrm{SL}(2,\mathbb{R})$ symmetry. When it comes to the $H_3^+$-Liouville relation, it is more convenient to use the following $\nu$-basis fields:

$$\Psi^\ell(\nu|w) = |\nu|^{\ell+1}\int_{\mathbb{R}} dt\ e^{i\nu t}\Psi^\ell(t|w)\ . \tag{2.24}$$

The relation with the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset and N=2 Liouville theory is more naturally expressed using the $m$-basis fields, which diagonalize the $t$-dilatations and $\nu$-dilatations:

$$\Psi^\ell_{m,\eta} = \int_{-\infty}^{\infty} dt\ |t|^{-\ell-1+m}\mathrm{sgn}^\eta(t)\Psi^\ell(t) \tag{2.25}$$

$$= N^\ell_{m,\eta}\int \frac{d\nu}{|\nu|}|\nu|^{-m}\mathrm{sgn}^\eta(\nu)\Psi^\ell(\nu)\,, \tag{2.26}$$

where physical values of $m$ are pure imaginary, and we define

$$\eta\in\{0,1\}\quad,\quad N^\ell_{m,\eta} = 2i^\eta\Gamma(-\ell+m)\sin\tfrac{\pi}{2}(-\ell-1+m-\eta)\ . \tag{2.27}$$

The boundary two-point function of open strings living on a single $AdS_2$ brane of parameter $r$ is known to be [10][1]

$$\left\langle \Psi^{\ell_1}(t_1|w_1)\Psi^{\ell_2}(t_2|w_2)\right\rangle_r = |w_{12}|^{-2\Delta_{\ell_1}} \times\frac{1}{2\pi}\left[\delta(\ell_1+\ell_2+1)\delta(t_{12}) + \delta(\ell_1-\ell_2)\tilde{R}^H_r(\ell_1)|t_{12}|^{2\ell_1}\right] \tag{2.28}$$

[1]Our formulas agree with [10] only up to renormalization of the boundary fields.

or equivalently

$$\left\langle \Psi^{\ell_1}(\nu_1|w_1)\Psi^{\ell_2}(\nu_2|w_2)\right\rangle_r = |w_{12}|^{-2\Delta_{\ell_1}} \\ \times |\nu_1|\delta(\nu_1+\nu_2)\left[\delta(\ell_1+\ell_2+1)+R^H_r(\ell_1)\delta(\ell_1-\ell_2)\right] \ . \quad (2.29)$$

The $H_3^+$ boundary "reflection number" $\tilde{R}^H_r(\ell)$ is related to the $H_3^+$ boundary reflection coefficient $R^H_r(\ell)$ by

$$\tilde{R}^H_r(\ell) = \frac{\pi}{\sin \pi\ell}\frac{1}{\Gamma(2\ell+1)}R^H_r(\ell) \ . \quad (2.30)$$

The quantity $R^H_r(\ell)$ deserves to be called the boundary reflection coefficient because of its rôle in the simple reflection property of the $\nu$-basis field,

$$\Psi^\ell(\nu|w) = R^H_r(\ell)\Psi^{-\ell-1}(\nu|w) \ . \quad (2.31)$$

Explicitly, $R^H_r(\ell)$ can be written in terms of the Liouville boundary reflection coefficient (C.3), provided the Liouville parameter is chosen as $b=(k-2)^{-1/2}$:

$$R^H_r(\ell) = R^L_{\frac{r}{2\pi b}-\frac{i}{4b},\frac{r}{2\pi b}+\frac{i}{4b}}\left(b(\ell+1)+\frac{1}{2b}\right) \ . \quad (2.32)$$

This relation between the $H_3^+$ and Liouville boundary reflection coefficients is not surprising given the relation $R^H(j) = R^L(b(j+1)+\frac{1}{2b})$ [9] between bulk reflection coefficients; the boundary relation actually follows from the relation between the boundary states of the $AdS_2$ brane in $H_3^+$ and the FZZT brane in Liouville theory [15], via the computation of the annulus amplitude.

Another known useful correlator is the bulk one-point function [10, 16]

$$\begin{aligned}\left\langle \Phi^j(x|z)\right\rangle_r &= \frac{1}{|z-\bar{z}|^{2\Delta_j}}\left[-\pi b^2\gamma(-b^2)\right]^{j+\frac{1}{2}}(8b^2)^{-\frac{1}{4}} \\ &\quad\times |x+\bar{x}|^{2j}\Gamma(1+b^2(2j+1))e^{-r(2j+1)\mathrm{sgn}(x+\bar{x})}, &(2.33)\\ \left\langle \Phi^j(\mu|z)\right\rangle_r &= \frac{1}{|z-\bar{z}|^{2\Delta_j}}\left[-\pi b^2\gamma(-b^2)\right]^{j+\frac{1}{2}}(8b^2)^{-\frac{1}{4}} \\ &\quad\times|\mu|\delta(\Re\mu)\Gamma(2j+1)\Gamma(1+b^2(2j+1))\cosh(2j+1)(r-i\tfrac{\pi}{2}\mathrm{sgn}\Im\mu) \ . &(2.34)\end{aligned}$$

## 3. $H_3^+$ correlators on a disc

Here we will study arbitrary $H_3^+$ correlators on a disc. We will express them in terms of Liouville correlators, which we consider as known quantities. The use of Liouville correlators will become natural after we recall that the Knizhnik-Zamolodchikov equations, which follow from the assumption that our $H_3^+$ correlators preserve the affine Lie algebra symmetry of the model, are equivalent to the Belavin-Polyakov-Zamolodchikov equations satisfied by certain Liouville correlators. Due to the existence of singularities, the KZ equations together with the usual factorization axioms are not enough for fully determining the $H_3^+$ correlators; we will introduce the additional assumption of continuity at the singularities.

Then we will exhibit a solution eq. (3.18) of all these requirements in terms of Liouville correlators. In the case of the bulk two-point function on the disc, we will prove that this solution is unique, even though our continuity assumption is weaker than the usual assumptions of the conformal bootstrap formalism.

## 3.1 Axioms for $H_3^+$ correlators on a disc

### 3.1.1 Symmetry requirements

We have already written the global $\mathrm{SL}(2,\mathbb{R})$ symmtry condition (2.23) for $H_3^+$ correlators on a disc. Here we concentrate on the KZ equations, which follow from the local $\widehat{s\ell_2}$ symmetry. It was shown in [15] that the gluing conditions for the $AdS_2$ branes are trivial in the $\mu$-basis, which implies that the disc correlators satisfiy the same KZ equations as the sphere correlators obtained by the "doubling trick",

$$\left\langle \prod_{a=1}^{n} \Phi^{j_a}(\mu_a|z_a) \prod_{b=1}^{m} \Psi^{\ell_b}(\nu_b|w_b) \right\rangle_{\text{disc}} \to \left\langle \prod_{a=1}^{n} \left(\Phi^{j_a}(\mu_a|z_a)\Phi^{j_a}(\bar{\mu}_a|\bar{z}_a)\right) \prod_{b=1}^{m} \Phi^{\ell_b}(\nu_b|w_b) \right\rangle_{\text{sphere}} \tag{3.1}$$

The KZ equations for a bulk correlator $\Omega_n^H = \langle \prod_{a=1}^n \Phi^{j_a}(\mu_a|z_a)\rangle$ are:

$$\left( (k-2)\frac{\partial}{\partial z_a} + \sum_{b\neq a} \frac{2t_a^3 t_b^3 - t_a^- t_b^+ - t_a^+ t_b^-}{z_a - z_b} \right) \Omega_n^H \;=\; 0\,, \begin{cases} t_a^+ = \mu_a \\ t_a^3 = \mu_a \frac{\partial}{\partial \mu_a} \\ t_a^- = \mu_a \frac{\partial^2}{\partial \mu_a^2} - \frac{j_a(j_a+1)}{\mu_a} \end{cases} . \tag{3.2}$$

The power of these equations comes from the fact that they are first order differential equations in $z_a$. So if we know a correlator at some value of $z_1$ or in some limit say $z_1 \to z_2$, then the $\frac{\partial}{\partial z_1}$ KZ equation determines that correlator for all values of $z_1$, provided no singularities are met on the way.

Explicit solutions of the KZ equations are known only in a few cases, some of which we will see in sections 4 and 5. For our present purposes, it will however be enough to solve the KZ equations in terms of Liouville correlators and conformal blocks. This is possible thanks to the KZ-BPZ relation [9, 17], which relates the KZ equations for our $H_3^+$ disc correlators to the BPZ equations satisfied by certain Liouville disc correlators. We will denote this as a relation $\simeq$ between $H_3^+$ and Liouville disc correlators. (The KZ and BPZ equations do not depend on the boundary conditions, which are therefore omitted in the following formula.)

$$\begin{aligned} \left\langle \prod_{a=1}^{n} \Phi^{j_a}(\mu_a|z_a) \prod_{b=1}^{m} \Psi^{\ell_b}(\nu_b|w_b) \right\rangle &\simeq \delta\left(2\textstyle\sum_{a=1}^n \Re\mu_a + \sum_{b=1}^m \nu_b\right)\ |u|\ |\Theta_{n,m}|^{\frac{k-2}{2}} \\ &\times \left\langle \prod_{a=1}^{n} V_{\alpha_a}(z_a) \prod_{b=1}^{m} B_{\beta_b}(w_b) \prod_{a'=1}^{n'} V_{-\frac{1}{2b}}(y_{a'}) \prod_{b'=1}^{m'} B_{-\frac{1}{2b}}(y_{b'}) \right\rangle , \end{aligned} \tag{3.3}$$

where the $H_3^+$ model at level $k$ is related to Liouville theory at parameter $b$, background charge $Q$ and central charge $c_L$ with

$$b^2 = \frac{1}{k-2} \quad , \quad Q = b + \frac{1}{b} \quad , \quad c_L = 1 + 6Q^2 \,. \tag{3.4}$$

The $H_3^+$ spins $j, \ell$ are related to Liouville momenta $\alpha, \beta$ as

$$\alpha = b(j+1) + \frac{1}{2b} \,, \ \beta = b(\ell+1) + \frac{1}{2b} \,. \tag{3.5}$$

The $n'$ bulk degenerate Liouville fields $V_{-\frac{1}{2b}}$ and $m'$ boundary fields $B_{-\frac{1}{2b}}$ are introduced at positions determined by Sklyanin's change of variables, which changes the isospin variables $\mu_a, \nu_b$ subject to the condition $2\sum_{a=1}^n \Re\mu_a + \sum_{b=1}^m \nu_b = 0$ (from global $s\ell(2)$ symmetry) into the variables $y_{a'}, y_{b'}$ defined as the $2n' + m' = 2n + m - 2$ zeroes of the function

$$\varphi(t) = \sum_{a=1}^n \frac{\mu_a}{t - z_a} + \sum_{a=1}^n \frac{\bar{\mu}_a}{t - \bar{z}_a} + \sum_{b=1}^m \frac{\nu_b}{t - w_b} \,, \tag{3.6}$$

plus one real variable

$$u = 2\sum_{a=1}^n \Re(\mu_a z_a) + \sum_{b=1}^m \nu_b w_b \,. \tag{3.7}$$

The prefactor $\Theta_{n,m}$ is written in terms of $Z_c = (z_a, \bar{z}_a, w_b)$ and $Y_d = (y_{a'}, \bar{y}_{a'}, y_{b'})$ as

$$\Theta_{n,m} = \frac{\prod_{c<c'\leq 2n+m}(Z_c - Z_{c'}) \prod_{d<d'\leq 2n+m-2}(Y_d - Y_{d'})}{\prod_{c=1}^{2n+m} \prod_{d=1}^{2n+m-2}(Z_c - Y_d)} \,. \tag{3.8}$$

We just provided enough data to make the relation (3.3) between KZ and BPZ equations explicit. Let us give more details on some relevant aspects and implications of this relation.

**A closer look at Sklyanin's separation of variables.** There is in general no explicit formula for the degenerate field positions $y$ as functions of the isospin variables $\mu, \nu$. However, the definition of $y$ as zeroes of a function $\varphi(t)$ (3.6) can be reformulated as

$$\varphi(t) = u \frac{\prod_{d=1}^{2n+m-2}(t - Y_d)}{\prod_{c=1}^{2n+m}(t - Z_c)} \,, \tag{3.9}$$

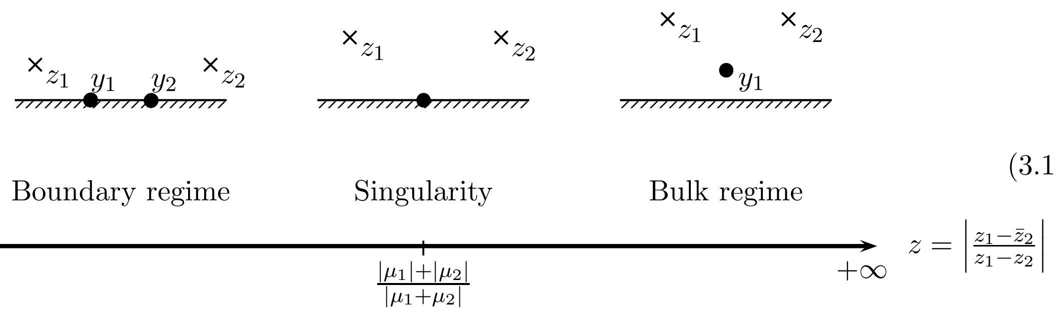
which by taking the limit $t \to z_a$ or $t \to w_b$ provides an explicit formula for $\mu_a$ or $\nu_b$ in terms of $y$:

$$\begin{aligned} \mu_a &= u \frac{\prod_{d=1}^{2n+m-2}(z_a - Y_d)}{(z_a - \bar{z}_a)\prod_{a'\neq a\leq n}(z_a - z_{a'})(z_a - \bar{z}_{a'})\prod_{b=1}^m (z_a - w_b)} \,, \\ \nu_b &= u \frac{\prod_{d=1}^{2n+m-2}(w_b - Y_d)}{\prod_{a=1}^n |w_b - z_a|^2 \prod_{b'\neq b\leq m}(w_b - w_{b'})} \,. \end{aligned} \tag{3.10}$$

**Singularities of KZ solutions.** The KZ-BPZ relation (3.3) allows us to easily study the singularities of the KZ solutions, because the Liouville correlators on the right hand-side are singular if and only if Liouville fields collide with each other or with the boundary. If such a collision involves only the fields $V_{\alpha_a}(z_a)$ and $B_{\beta_b}(w_b)$, then the corresponding singularity at $z_a = z_{a'}, z_a = \bar{z}_a$ or $w_b = w_{b'}$ is the power-like singularity expected from the $H_3^+$ model on general grounds.

However, extra singularities occur where degenerate Liouville fields $V_{-\frac{1}{2b}}(y_{a'})$ (or $B_{-\frac{1}{2b}}(y_{b'})$) are involved. If such a degenerate field comes close to $V_{\alpha_a}(z_a)$ (or $B_{\beta_b}(w_b)$), then $\varphi(t)$ loses its pole at $t = z_a$ which implies $\mu_a = 0$ (respectively, $\nu_b = 0$). Such singularities will play no significant rôle in the following, and should be considered as artefacts of the $\mu$-basis. On the other hand, singularities arising from collision of two boundary degenerate fields to become one bulk degenerate field $B_{-\frac{1}{2b}} B_{-\frac{1}{2b}} \to V_{-\frac{1}{2b}}$ (or vice versa) will play a crucial rôle;[2] in the following we will always refer to these singularities when writing about singularities of $H_3^+$ correlators. Let us explain their importance in the case of the bulk two-point function on the disc $\left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\right\rangle$. (This case was already studied in [15].)

Given $\sum_{a=1}^2 \Re\mu_a = 0$, the function $\varphi(t) = \sum_{a=1}^2 \left(\frac{\mu_a}{t-z_a} + \frac{\bar{\mu}_a}{t-\bar{z}_a}\right)$ has two zeroes. If they are both real, they correspond to two Liouville degenerate boundary fields in a correlator $\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)B_{-\frac{1}{2b}}(y_1)B_{-\frac{1}{2b}}(y_2)\right\rangle$: we call this situation the *boundary regime*. If they are complex conjugate, they define the position of one Liouville degenerate bulk field in a correlator $\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{-\frac{1}{2b}}(y_1)\right\rangle$: we call this the *bulk regime*. The positions of the Liouville fields involved in the KZ-BPZ relation (3.3) in the case of the $H_3^+$ bulk two-point function on the disc can be depicted as:




(3.11)

This singularity is significant because it separates two regimes which are not otherwise connected, since the cross-ratio $z$ takes real values. This is in contrast to the similar singularity which appears in the $H_3^+$ four-point function on a sphere. The related Liouville correlator is in that case $\left\langle \prod_{a=1}^4 V_{\alpha_a}(z_a)\ V_{-\frac{1}{2b}}(y_1)V_{-\frac{1}{2b}}(y_2)\right\rangle$, and one can go around the singularity $y_1 = y_2$ by moving $y_1, y_2$ in the Riemann sphere.

### 3.1.2 Factorization axioms

Factorization is a standard axiom of quantum field theory. It states that in the limit where two of the fields come close, the correlator $\left\langle \prod_{a=1}^n \Phi^{j_a}(\mu_a|z_a) \prod_{b=1}^m \Psi^{\ell_b}(\nu_b|w_b)\right\rangle$ reduces to lower correlators determined by the operator product expansion of the two fields. We of course assume that $H_3^+$ correlators obey such factorization axioms. Note that factorization will only require taking limits of the worldsheet positions $z_a, w_b$ of the fields, while their isospin variables $\mu_a, \nu_b$ are kept fixed and arbitrary.

[2] Such singularities are presumably equivalent to the "$z = x$" singularity in Fateev and Zamolodchikov's KZ-BPZ relation [18] in the $x$-basis.



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Depending on the nature of the two fields which come close, there are three types of factorization, which correspond to inserting the three types of operator product expansions into the correlators:

- Bulk OPE:

$$\Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2) \underset{z_{12}\to 0}{\sim} \int dj \int \frac{d^2\mu}{|\mu|^2}|z-z_1|^{4\Delta_j} \\ \left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\Phi^{-j-1}(-\mu|z)\right\rangle \times \\ \times \left(\Phi^j(\mu|z_1) + O(z_{12})\right), \tag{3.12}$$

- Bulk-boundary OPE:

$$\Phi^j(\mu|z) \underset{z-\bar{z}\to 0}{\sim} \int d\ell \int \frac{d\nu}{|\nu|}|w-z|^{2\Delta_\ell} \\ \left\langle \Phi^j(\mu|z)\Psi^{-\ell-1}(-\nu|w)\right\rangle_r \times \left({}_r\Psi^\ell(\nu|z)_r + O(z-\bar{z})\right), \tag{3.13}$$

- Boundary OPE:

$${}_{r_1}\Psi^{\ell_1}(\nu_1|w_1)_r\Psi^{\ell_2}(\nu_2|w_2)_{r_2} \underset{w_{12}\to 0}{\sim} \int d\ell \, \frac{d\nu}{|\nu|}|w-w_1|^{2\Delta_\ell} \\ \left\langle {}_{r_1}\Psi^{\ell_1}(\nu_1|w_1)_r\Psi^{\ell_2}(\nu_2|w_2)_{r_2}\Psi^{-\ell-1}(-\nu|w)_{r_1}\right\rangle \times \\ \times \left({}_{r_1}\Psi^\ell(\nu|w_1)_{r_2} + O(w_{12})\right) . \tag{3.14}$$

(Note that the OPEs do not depend on the choice of the auxiliary worldsheet variables $z, w$.)

We can formally write these OPEs without knowing the three basic correlators (bulk three-point, bulk-boundary two-point, boundary three-point functions); on the other hand we rely on the previous knowledge of the bulk and boundary spectra and two-point functions[3] eq. (2.13), (2.29).

Once inserted into a correlator, such an OPE should be considered as a formal limit, since the corrections $O(z_{12})$ to one term $j$ can be dominant with respect to the leading contribution $\Phi^{j'}(\mu|z_1)$ of another term $j'$ of higher conformal dimension. This formal limit consists in focussing on the contribution of primary fields, and the corrections correspond to descendants. Such corrections are in principle determined by the symmetry of the model, in our case the affine Lie algebra symmetry.

**Factorization and Cardy-Lewellen formalism.** We now discuss the crucial issue of the strength of the factorization constraints, i.e. in which measure they determine the correlators. First note that if the sum over all descendant contributions converged for any values of the worldsheet variables, then the correlators would be fully determined by their

[3] Although we do not yet know the spectrum of boundary fields ${}_{r_1}\Psi^\ell(\nu|w)_{r_2}$ when $r_1 \neq r_2$, we assume that such fields are parametrized by the same values of $\ell$ and $\nu$ as in the case $r_1 = r_2$, and do not have additional indices.

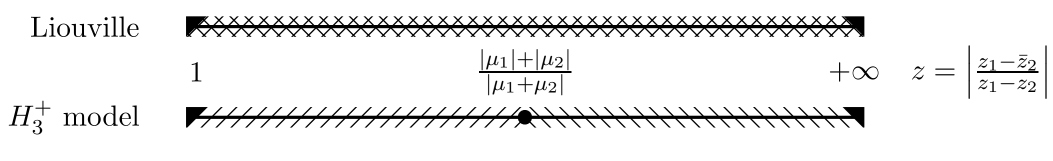
behaviour in one given factorization limit. For example, we would fully know the bulk two-point function on the disc thanks to the limit where it reduces to the known bulk three-point function on a sphere and bulk one-point function on the disc:

$$\begin{aligned}\left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\right\rangle_r \underset{z_{12}\to 0}{\sim} & \int dj \int \frac{d^2\mu}{|\mu|^2}|z-z_1|^{4\Delta_j} \\ & \left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\Phi^{-j-1}(-\mu|z)\right\rangle \times \\ & \times \left(\left\langle \Phi^{j}(\mu|z_1)\right\rangle_r + O(z_{12})\right) \end{aligned} \tag{3.15}$$

We could now study $\left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\right\rangle_r$ in the limit $z_1 \to \bar{z}_1$. Whether it would factorize or not would be a consistency test on the bulk three-point and disc one-point functions. If the test was passed, we could then deduce the bulk-boundary two-point function. Such constraints and relations for structure constants were systematically studied by Cardy and Lewellen [7, 8].

The Cardy-Lewellen formalism actually applies in the cases of Liouville theory and of the $H_3^+$ model on the sphere. In the latter case, the sums of descendant contributions however do not converge for all values of the worldsheet variables $z$ (as is apparent from the existence of singularities), but only in neighbourhoods of the various factorization limits. But the affine Lie algebra symmetry which in principle determines these sums actually yields a more powerful tool: the KZ equations. These equations can be used to analytically continue the correlators in regions where the sums of descendants do not converge.

On the disc however, the $H_3^+$ bulk two-point function is not fully determined by its behaviour near $z_{12} \to 0$, because as shown in the picture (3.11) it is impossible to go around the singularity. We would need as additional data the behaviour near $z_1 - \bar{z}_1 \to 0$, and therefore the (as yet unknown) bulk-boundary two-point function.[4] In terms of sums of descendants, the situation is presumably the following: the sum of descendants in the bulk-boundary OPE converges near $z_1 = \bar{z}_1$ and in the vicinity (up to the singularity), and therefore in the boundary regime. The sum of descendants in the bulk OPE converges near $z_1 = z_2$ and in the vicinity (up to the singularity), and therefore in the bulk regime. But the strength of the Cardy-Lewellen constraints relies on the existence of an overlap between the domains of convergences of these two OPEs. Such an overlap is absent in our case, as opposed to the case of the bulk two-point function on the disc in Liouville theory, where the sums of descendants in both OPEs converge for any values of $z_1, z_2$ as was established in [19] (section 2.4 therein):




[4]The situation is even worse in the case of the boundary four-point function $\left\langle \prod_{b=1}^4 \Psi^{\ell_b}(\nu_b|w_b)\right\rangle$: even if we knew the boundary three-point function and therefore the behaviour in both possible factorization limits $w_{12} \to 0, w_{23} \to 0$, we could not deduce the boundary four-point function in the regime where the corresponding Liouville correlator has one bulk degenerate field.

(Here the triangles denote the factorization limits, and the hatches the corresponding regions where the sums of descendants converge.)

In this sense, the Cardy-Lewellen formalism does not fully apply to the $H_3^+$ model on the disc because of the singularities of the $H_3^+$ correlators. Nevertheless, we can recover part of the power of the Cardy-Lewellen constraints by making a natural assumption on the behaviour of the $H_3^+$ correlators at the singularities.

#### 3.1.3 Continuity assumption

In contrast to the symmetry requirements and factorization axioms, which are standard assumptions of conformal field theory in the conformal bootstrap formalism, our continuity assumption will be a novelty of the $H_3^+$ model on the disc. Such an assumption is made necessary by the existence of extra singularities of the model (3.11): for the formalism to be of any use, we need some control over the behaviour of correlators at these singularities.

**Continuity assumption:** The $H_3^+$ correlators are continuous at the singularities which occur when degenerate fields in the corresponding Liouville correlators collide.

In order to clarify the meaning of this assumption, let us recall how KZ solutions behave near such singularities. This can easily be deduced from the relevant Liouville OPEs, dressed with the $|y_{12}|^{\frac{k-2}{2}}$ prefactor from the KZ-BPZ relation (3.3),

$$|y_{12}|^{\frac{k-2}{2}} B_{-\frac{1}{2b}}(y_1) B_{-\frac{1}{2b}}(y_2) \underset{y_{12}\to 0}{\sim} B_{-\frac{1}{b}}(y_1) + C^L(-\frac{1}{2b}, -\frac{1}{2b}, Q) |y_{12}|^{2k-3} B_0(y_1) \tag{3.16}$$

$$|y_{12}|^{\frac{k-2}{2}} V_{-\frac{1}{2b}}(y_1) \underset{y_1-\bar{y}_1\to 0}{\sim} B_{-\frac{1}{b}}(y_1) + B^L(-\frac{1}{2b}, Q) |y_{12}|^{2k-3} B_0(y_1)\,, \tag{3.17}$$

where we omit the dependences on the boundary parameters of the Liouville boundary three-point function $C^L(-\frac{1}{2b}, -\frac{1}{2b}, Q)$, bulk-boundary two-point function $B^L(-\frac{1}{2b}, Q)$, and boundary fields. (Explicit formulas for the relevant OPE coefficients can be found in the appendix, eq. (C.7) and (C.5).)

The leading behaviour of the KZ solutions therefore consists of two terms, associated with the Liouville boundary fields $B_0$ and $B_{-\frac{1}{b}}$. (The corrections to the leading behaviour are due to descendants of these two fields.) The critical exponent of the $B_{-\frac{1}{b}}$ term is zero, so such a term has a finite limit whether it arises from the bulk regime ($V_{-\frac{1}{2b}}$ case) or from the boundary regime ($B_{-\frac{1}{2b}} B_{-\frac{1}{2b}}$ case). The critical exponent of the $B_0$ term is $2k-3 > 1$, such a term goes to zero at the singularity. Therefore, all KZ solutions have finite limits at the singularity. Our continuity assumption means that the limit evaluated from the bulk regime should agree with the limit evaluated from the boundary regime. This seems to us a very natural assumption.

Thus, the continuity assumption will be a nontrivial requirement on $H_3^+$ correlators, although it of course does not fully determine how KZ solutions behave through the singularity, because the $B_0$ term remains unconstrained.

### 3.2 $H_3^+$ disc correlators from Liouville theory

It is relatively easy to find an Ansatz for the $H_3^+$ disc correlators which satisfies all our axioms. The difficulty will be to prove that the solution is unique. Let us first write our

Ansatz for arbitrary $H_3^+$ correlators on the disc:

$$\begin{aligned}&\left\langle \prod_{a=1}^{n} \Phi^{j_a}(\mu_a|z_a) \prod_{b=1}^{m} {}_{r_{b-1,b}}\Psi^{\ell_b}(\nu_b|w_b)_{r_{b,b+1}} \right\rangle \\ &\qquad = \pi^2\sqrt{\tfrac{b}{2}}(-\pi)^{-n}\ \delta\left(2\textstyle\sum_{a=1}^{n}\Re\mu_a + \sum_{b=1}^{m}\nu_b\right)\ |u|\ |\Theta_{n,m}|^{\frac{k-2}{2}} \\ &\qquad\quad \times \left\langle \prod_{a=1}^{n} V_{\alpha_a}(z_a) \prod_{b=1}^{m} {}_{s_{b-1,b}}B_{\beta_b}(w_b)_{s_{b,b+1}} \prod_{a'=1}^{n'} V_{-\frac{1}{2b}}(y_{a'}) \prod_{b'=1}^{m'} B_{-\frac{1}{2b}}(y_{b'}) \right\rangle , \end{aligned} \tag{3.18}$$

where most notations were already defined in our study of the KZ-BPZ relation: the Liouville parameter $b$ (3.4), the Liouville momenta $\alpha, \beta$ (3.5), the quantity $u$ (3.7), the prefactor $\Theta_{n,m}$ (3.8). The positions $y_{a'}, y_{b'}$ of the Liouville degenerate fields were defined as the zeroes of a function $\varphi(t)$ (3.6). In addition, we specify the set of boundary conditions $s_{b-1,b}$ by

$$s = \frac{r}{2\pi b} - \frac{i}{4b}\mathrm{sgn}\varphi(t)\ . \tag{3.19}$$

That is, the Liouville boundary parameter $s$ on a point $t$ of the boundary is given by the $H_3^+$ boundary parameter $r$, shifted by a quantity which depends on $\mathrm{sgn}\varphi(t)$. (Indeed $\varphi(t)$ is real if $t$ is real.) Notice that $\varphi(t)$ changes sign at its zeroes, which are the positions of the boundary degenerate fields, and when it is infinite, which happens at the points where the generic boundary fields ${}_{s_{b-1,b}}B_{\beta_b}(w_b)_{s_{b,b+1}}$ are inserted. So each boundary degenerate field $B_{-\frac{1}{2b}}(y_{b'})$ induces a jump $\pm\frac{i}{2b}$ of the boundary parameter $s$, consistently with the results of Fateev, Zamolodchikov and Zamolodchikov [11]. Then, for a given $AdS_2$ brane parameter $r$, there correspond two opposite values of the Liouville boundary cosmological constant,

$$\mu_B = \sqrt{\frac{\mu_L}{\sin\pi b^2}}\cosh 2\pi b s = \pm\sqrt{\frac{\mu_L}{\sin\pi b^2}}\sinh r\ . \tag{3.20}$$

The formula (3.18) is our main result and the rest of the article is devoted to giving evidence for it, and drawing some consequences.

The first check is the compatibility with the bulk one-point function, which is explicitly known (2.34). This check is straighforward and was already performed in [15].

Let us check that our formula satisfies the axioms of the $H_3^+$ model. By construction, our Ansatz (3.18) satisfies the KZ equations. It is continuous at the singularities due to the agreement between the coefficients of the leading terms of the Liouville $\lim_{y_{12}\to 0} B_{-\frac{1}{2b}}(y_1)B_{-\frac{1}{2b}}(y_2)$ and $\lim_{z_1-\bar{z}_1\to 0} V_{-\frac{1}{2b}}(z_1)$ OPEs (3.16), (3.17). The only subtle issues come from the factorization axioms:

- Bulk factorization $z_{12}\to 0$: the pole $t = z_1$ of the function $\varphi(t)$ (3.6) must remain simple, so that one Liouville bulk degenerate field say $V_{-\frac{1}{2b}}(y_1)$ must come close to $V_{\alpha_1}(z_1)$ and $V_{\alpha_2}(z_2)$, i.e. $y_1 - z_1 \propto z_{12} \to 0$. Thus, we should insert into our

Ansatz (3.18) the following Liouville OPE:

$$V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{-\frac{1}{2b}}(y_1) \underset{z_{12}\propto z_1-y_1\to 0}{\sim} \int d\alpha\ |z-z_1|^{4\Delta_\alpha} \\ \left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{-\frac{1}{2b}}(y_1)V_{Q-\alpha}(z)\right\rangle \times \left(V_\alpha(z_1)+\mathcal{O}(z_{12})\right)\ , \quad (3.21)$$

where $\Delta_\alpha = \alpha(Q-\alpha)$ is the conformal dimension of a Liouville field of momentum $\alpha$. This is the crucial step in proving that our Ansatz indeed satisfies the bulk OPE axiom (3.12), as was shown in detail in [9] in the case of $H_3^+$ correlators on the sphere.

- Bulk-boundary factorization $z_1 - \bar{z}_1 \to 0$: by a similar reasoning, one Liouville boundary degenerate field say $B_{-\frac{1}{2b}}(y_1)$ must come close to $V_{\alpha_1}(z_1)$. We should insert into our Ansatz (3.18) the following Liouville OPE:

$$V_{\alpha_1}(z_1)B_{-\frac{1}{2b}}(y_1) \underset{z_1-\bar{z}_1\propto z_1-y_1\to 0}{\sim} \int d\beta\ |w-z_1|^{2\Delta_\beta} \\ \left\langle V_{\alpha_1}(z_1)B_{-\frac{1}{2b}}(y_1)_{s_-}B_{Q-\beta}(w)_{s_+}\right\rangle \times \left({}_{s_-}B_\beta(z_1)_{s_+}+\mathcal{O}(z_1-\bar{z}_1)\right)\ , \quad (3.22)$$

with $s_\pm = \frac{r}{2\pi b} \pm \frac{i}{4b}\mathrm{sgn}(\mu_1+\bar{\mu}_1)$, and $r$ is the $H_3^+$ boundary parameter at the point where $z_1$ reaches the boundary. Then one can check that the Liouville correlator $\left\langle V_{\alpha_1}(z_1)B_{-\frac{1}{2b}}(y_1)_{s_-}B_{Q-\beta}(z_1)_{s_+}\right\rangle$ agrees with the prediction of our Ansatz (3.18) for the $H_3^+$ bulk-boundary two-point function appearing in the $H_3^+$ bulk-boundary OPE (3.13).

- Boundary factorization $w_{12} \to 0$: by a similar reasoning, one Liouville boundary degenerate field say $B_{-\frac{1}{2b}}(y_1)$ must come close to $B_{\beta_1}(w_1), B_{\beta_2}(w_2)$. We should insert into our Ansatz (3.18) the following Liouville OPE:

$$B_{\beta_1}(w_1)B_{\beta_2}(w_2)B_{-\frac{1}{2b}}(y_1) \underset{w_{12}\propto y_1-w_1\to 0}{\sim} \int d\beta\ |w-w_1|^{2\Delta_\beta} \\ \left\langle B_{\beta_1}(w_1)B_{\beta_2}(w_2)B_{-\frac{1}{2b}}(y_1)B_{Q-\beta}(w)\right\rangle \times \left(B_\beta(w_1)+\mathcal{O}(w_{12})\right)\ , \quad (3.23)$$

where for definiteness we assumed the degenerate field to come on the right on $B_{\beta_1}(w_1)$ and $B_{\beta_2}(w_2)$, while it may also come on the left or in between, depending on the signs of $\nu_1, \nu_2$ and $\nu_1+\nu_2$. For simplicity, we omit the Liouville boundary parameters, which can easily be deduced from our Ansatz. This is the main step in checking that our Ansatz (3.18) is compatible with the $H_3^+$ boundary OPE (3.14).

There is however a property which we have not checked: the $\mathrm{SL}(2,\mathbb{R})$ group symmetry (2.23), or equivalently its Lie algebra version $s\ell(2,\mathbb{R})$. In the absence of boundaries, this symmetry is necessary for the KZ-BPZ relation [14, 9], and is therefore automatically included in the $H_3^+$-Liouville relation. However, it is not obvious that our Ansatz is $s\ell(2,\mathbb{R})$ symmetric, because the Liouville boundary parameter (3.19) varies along the boundary, in a way which is non-trivially affected by $s\ell(2,\mathbb{R})$ transformations. In the case of the bulk-boundary two-point function (section 5), we will explicitly check the $\mathrm{SL}(2,\mathbb{R})$ symmetry of our Ansatz.



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### 3.3 Uniqueness of the solution to the axioms

We have easily checked that our formula (3.18) for the $H_3^+$ disc correlators verifies our axioms of symmetry, factorization and continuity. We will now argue that this solution is unique in the particular case of correlators with no boundary condition changing operators.

We will write an explicit argument only in the case of the bulk two-point function on the disc. This will be enough to address the crucial issue of the singularity separating the bulk and boundary regimes, as defined in (3.11). Let us spell out the formula to be proved:

$$\left\langle \Phi^{j_1}(\mu_1|z_1)\Phi^{j_2}(\mu_2|z_2)\right\rangle_r = \sqrt{\tfrac{b}{8}}\delta\left(\Re(\mu_1+\mu_2)\right)\ |u|\ \left(\frac{|z_{12}|^2|y_{12}|\prod_a|z_a-\bar{z}_a|^2}{\prod_{a,b}|z_a-y_b|^2}\right)^{\frac{k-2}{2}}$$
$$\times\begin{cases}\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{-\frac{1}{2b}}(y_1)\right\rangle_{s_+} & \text{if } y_2=\bar{y}_1 \text{ (bulk regime)}\,,\\ \left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)_{s_+}B_{-\frac{1}{2b}}(y_1)_{s_-}B_{-\frac{1}{2b}}(y_2)_{s_+}\right\rangle & \text{if } y_1<y_2\in\mathbb{R} \text{ (boundary regime)}\,,\end{cases} \tag{3.24}$$

where $s_\pm = \frac{r}{2\pi b}\mp\frac{i}{4b}\mathrm{sgn}u$ with $u=2\Re(\mu_1z_1+\mu_2z_2)$, and in the bulk regime we have $\mathrm{sgn}u=\mathrm{sgn}\Im(\mu_1+\mu_2)$.

The explicit knowledge of the $H_3^+$ bulk one-point function on the disc, and the axiom of bulk factorization (3.12), are enough to prove the formula (3.24) in the limit $z_{12}\to 0$. Then, the local $\widehat{s\ell_2}$ symmetry requirement and the knowledge that the resulting KZ equations are equivalent to BPZ equations (3.3) show that the formula is true in the whole bulk regime.

The continuity assumption will now provide some information on the bulk two-point function at the $z=\frac{|\mu_1|+|\mu_2|}{|\mu_1+\mu_2|}$ end of the boundary regime. The other end $z=1$ is constrained by the axiom of bulk-boundary factorization (3.13), which is a non-trivial requirement even though we do not know the bulk-boundary two-point function. These two limiting regions are connected by the KZ equations, which hold in the whole boundary regime. We purport to show that, taken toghether, these constraints are enough to fully determine the bulk two-point function in the boundary regime.

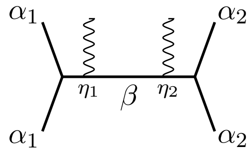
The reasoning could now go in two possible directions, depending on which one of the two limiting regions we consider first. If we first solve the continuity assumption, it is then difficult to exploit the axiom of bulk-boundary factorization. So we will first solve the latter axiom.

**Solving the axiom of bulk-boundary factorization.** We will write the general solution of this axiom in terms of some arbitrary structure constants $B_{r,\eta}(j,\ell)$, and $H_3^+$ conformal blocks built from known Liouville theory conformal blocks. The relevant conformal blocks are most easily defined by decomposing the boundary regime Ansatz (3.24),

$$\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)_{s_+}B_{-\frac{1}{2b}}(y_1)_{s_-}B_{-\frac{1}{2b}}(y_2)_{s_+}\right\rangle =$$
$$\sum_{\eta_1,\eta_2=\pm}\int d\beta\ B^L_{s_+}(\alpha_1,\beta-\tfrac{\eta_1}{2b})C^L_{s_+}(Q-\beta+\tfrac{\eta_1}{2b},-\tfrac{1}{2b}\underset{s_-}{|}\beta)B^L_{s_+}(\alpha_2,\beta-\tfrac{\eta_2}{2b})C^L_{s_+}(Q-\beta+\tfrac{\eta_2}{2b},-\tfrac{1}{2b}\underset{s_-}{|}\beta)$$
$$\times\left(R^L_{s_-,s_+}(\beta)\right)^{-1}\mathcal{G}_{\beta,\eta_1,\eta_2}(\alpha_1,\alpha_2|z_1,z_2,y_1,y_2)\ . \tag{3.25}$$

A basis of solutions of the Knizhnik-Zamolodchikov equations in the boundary regime is obtained by multiplying the conformal blocks $\mathcal{G}_{\beta,\eta_1,\eta_2}(\alpha_1,\alpha_2|z_1,z_2,y_1,y_2)$ with the prefactor (first line) of (3.24), while assuming the relation (3.5) between $H_3^+$ spins and Liouville momenta. We will still denote the resulting $H_3^+$ conformal blocks as $\mathcal{G}_{\beta,\eta_1,\eta_2}(\alpha_1,\alpha_2|z_1,z_2,y_1,y_2)$, and represent them schematically as

$$\mathcal{G}_{\beta,\eta_1,\eta_2}(\alpha_1,\alpha_2|z_1,z_2,y_1,y_2) =$$




, (3.26)

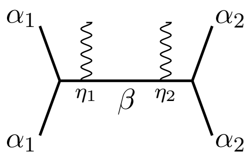
where the wiggly lines denote degenerate fields of momentum $-\frac{1}{2b}$, and the discrete indices $\eta_i = \pm$ indicate the fusion channels $\beta - \frac{\eta_i}{2b}$ of these degenerate boundary fields $B_{-\frac{1}{2b}}$ with another boundary field $B_\beta$.

The general solution of the bulk-boundary factorization axiom is obtained by replacing the Liouville structure constants $B^L_{s_+} C^L_{s_+}$ in eq. (3.25) with arbitrary quantities $B_{r,\eta}(j,\ell)$,

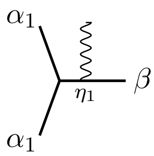
$$\mathcal{S} = \sum_{\eta_1,\eta_2=\pm} \int d\beta \; B_{r,\eta_1}(j_1,\ell) B_{r,\eta_2}(j_2,\ell) \left(R^H_r(\ell)\right)^{-1} \mathcal{G}_{\beta,\eta_1,\eta_2}(\alpha_1,\alpha_2|z_1,z_2,y_1,y_2) \; . \tag{3.27}$$

(Recall the relation (2.32) between the Liouville and $H_3^+$ boundary reflection coefficients.) We have indeed chosen our basis of conformal blocks for its factorizing behaviour in the boundary factorization limit,

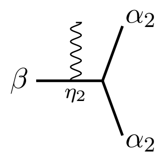
$$\lim_{z_1-\bar{z}_1\to 0}$$




$$= |w - z_1|^{2\Delta_\beta}$$




$\times$




, (3.28)

where the two factors depend on $\beta, \eta_1, j_1, z_1, y_1, w$ and $\beta, \eta_2, j_2, z_2, y_2, w$ respectively. Here $w$ is the position of the intermediate channel field of momentum $\beta$ on the boundary of the disc.

The quantities $B_{r,\eta}(j,\ell)$ can be interpreted as the bulk-boundary structure constants of the $H_3^+$ model. For given values of the bulk and boundary spins $j$ and $\ell$, there are *two* such structure constants labelled by $\eta = \pm$. The reason for this fact, and a detailed analysis of the $H_3^+$ bulk-boundary two-point function, are given in section 5.

Therefore, thanks to the bulk-boundary factorization axiom, our task is now reduced to determining the structure constants $B_{r,\eta}(j,\ell)$, i.e. showing that they agree with the Liouville structure constants in eq. (3.25). For this, we need the continuity assumption.

**Solving the continuity assumption.** We recall that the continuity assumption determines the terms which involve the $-\frac{1}{b}$ channel in the fusion product of the two boundary degenerate fields (3.16). In order to exploit this assumption, it is therefore convenient to use

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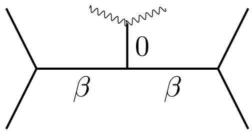
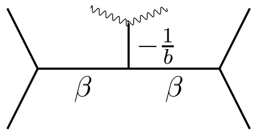
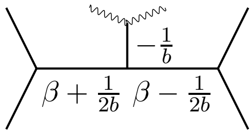
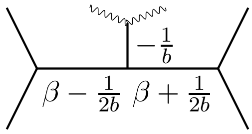
a new basis of conformal blocks (where we omit the dependence on $(\alpha_1, \alpha_2|z_1, z_2, y_1, y_2)$):

$\mathcal{G}_{\beta,0} =$ 

, $\mathcal{G}_{\beta,-\frac{1}{b},0} =$ 

,

$\mathcal{G}_{\beta,-\frac{1}{b},+} =$ 

, $\mathcal{G}_{\beta,-\frac{1}{b},-} =$ 

.(3.29)

The relation to our previous basis of conformal blocks is

$$\mathcal{G}_{\beta,\eta,\eta} = F_{\eta,0}(\beta)\ \mathcal{G}_{\beta+\frac{\eta}{2b},0} + F_{\eta,-\frac{1}{b}}(\beta)\ \mathcal{G}_{\beta+\frac{\eta}{2b},-\frac{1}{b},0} \quad , \quad \mathcal{G}_{\beta,\eta,-\eta} = \mathcal{G}_{\beta,-\frac{1}{b},-\eta}\,, \tag{3.30}$$

for some Liouville fusing matrix elements $F_{\eta,0}(\beta), F_{\eta,-\frac{1}{b}}(\beta)$ which depend on $\beta$ but not on $\alpha_1, \alpha_2$. (These fusing matrix elements are known explicitly, but we do not need their precise form.)

Let us rewrite the solution of the factorization axiom (3.27) in terms of such conformal blocks:

$$\begin{aligned}\mathcal{S} = \int d\beta\ \left(R_r^H(\ell)\right)^{-1} &\Big(B_{r,+}(j_1,\ell)B_{r,-}(j_2,\ell)\mathcal{G}_{\beta,-\frac{1}{b},-} + B_{r,-}(j_1,\ell)B_{r,+}(j_2,\ell)\mathcal{G}_{\beta,-\frac{1}{b},+}\Big) \\ +\sum_\eta \int d\beta\ \left(R_r^H(\ell)\right)^{-1} &B_{r,\eta}(j_1,\ell)B_{r,\eta}(j_2,\ell)\left(F_{\eta,0}(\beta)\ \mathcal{G}_{\beta+\frac{\eta}{2b},0}+F_{\eta,-\frac{1}{b}}(\beta)\ \mathcal{G}_{\beta+\frac{\eta}{2b},-\frac{1}{b},0}\right).\end{aligned} \tag{3.31}$$

The continuity assumption determines the terms in $\mathcal{G}_{\beta,-\frac{1}{b},\pm}$, and therefore the values of the products $B_{r,+}(j_1,\ell)B_{r,-}(j_2,\ell)$ and $B_{r,-}(j_1,\ell)B_{r,+}(j_2,\ell)$. All our conformal blocks are indeed linearly independent, up to the identity of blocks labelled by momenta with identical conformal weights, for instance $\mathcal{G}_{\beta,-\frac{1}{b},+} = \mathcal{G}_{Q-\beta,-\frac{1}{b},-}$. One should also take into account corresponding identities among the structure constants, namely $B_{r,\eta}(j,\ell) = R_r^H(\ell)B_{r,-\eta}(j,-\ell-1)$.

The resulting values of $B_{r,+}(j_1,\ell)B_{r,-}(j_2,\ell)$ and $B_{r,-}(j_1,\ell)B_{r,+}(j_2,\ell)$ must be the ones appearing in the decomposition of our Ansatz (3.25), because we already know the Ansatz to be a solution of the continuity constraints. This determines $B_{r,\pm}(j,\ell)$ up to a $j$-independent rescaling, $B_{r,\pm}(j,\ell) \to f_r(\ell)^{\pm 1}B_{r,\pm}(j,\ell)$. A non-trivial rescaling ($f_r(\ell) \neq \pm 1$) can however be excluded by exploiting the terms in $\mathcal{G}_{\beta+\frac{\eta}{2b},-\frac{1}{b},0}$, which are again determined by the continuity constraint. This shows that the Ansatz is the only solution to our axioms.

Therefore, our lack of control over the $\mathcal{G}_{\beta+\frac{\eta}{2b},0}$ terms has not prevented us from fully determining the bulk two-point function, thanks to the bulk-boundary factorization axiom. In the standard Cardy-Lewellen formalism, the bulk two-point function would be fully determined from the disc one-point and sphere three-point functions, and the bulk-boundary factorization axiom would then come as a consistency check on these quantities. In our case, this consistency check is weaker, because it can involve only the part of the axiom which we do not use for determining the bulk two-point function.

**Generalization.** This reasoning can be generalized to arbitrary $H_3^+$ bulk correlators on the disc. Indeed, the existence of a bulk regime where the $H_3^+$ correlators are known (thanks to the bulk OPE) gives a nontrivial content to the continuity assumption. Moreover, our determination of the $H_3^+$ bulk two-point function also yields the knowledge of the $H_3^+$ bulk-boundary two-point function. Therefore, we can in principle apply the bulk-boundary OPE (3.13) to arbitrary $H_3^+$ bulk correlators, which proves our main result (3.18) for correlators of bulk fields and boundary fields ${}_r\Psi^\ell_r$ which preserve the boundary condition. Boundary condition changing operators ${}_r\Psi^\ell_{r'}$ are more challenging: we leave their case as a conjecture, which is supported by our check of all the axioms, and the analysis of the boundary two-point function in section 4.

### 3.4 $H_3^+$-Liouville relation in the $m$-basis

The $m$-basis relation may be useful for the study of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model, which is formally quite close to the $H_3^+$-model in the $m$ basis. The relation is obtained by straightforward application of the integral transforms (2.11), (2.26) to the $\mu$-basis result (3.18):

$$\begin{aligned}&\left\langle \prod_{a=1}^n \Phi^{j_a}_{m_a,\bar{m}_a}(z_a) \prod_{b=1}^m \Psi^{\ell_b}_{m_b,\eta_b}(w_b)\right\rangle \\ &\quad\propto \prod_{a=1}^n N^{j_a}_{m_a,\bar{m}_a} \prod_{b=1}^m N^{\ell_b}_{m_b,\eta_b} \int_{\mathbb{R}} \frac{du}{|u|} \int d^{2n'+m'}y \prod_{a=1}^n \left(\mu_a^{-m_a}\bar{\mu}_a^{-\bar{m}_a}\right) \prod_{b=1}^m |\nu_b|^{-m_b} \mathrm{sgn}^{\eta_b}(\nu_b) \\ &\quad\times |\Theta_{n,m}|^{\frac{k}{2}} \left\langle \prod_{a=1}^n V_{\alpha_a}(z_a) \prod_{b=1}^m B_{\beta_b}(w_b) \prod_{a'=1}^{n'} V_{-\frac{1}{2b}}(y_{a'}) \prod_{b'=1}^{m'} B_{-\frac{1}{2b}}(y_{b'})\right\rangle \,. \end{aligned} \tag{3.32}$$

The non-trivial content of the formula is the fact that the Jacobian for Sklyanin's separation of variables (3.10) (which gives $\mu_a, \nu_b$ as a function of the positions $y_{a'}, y_{b'}$ of the degenerate fields) is $|u|^{-2}|\Theta_{n,m}| \prod_{a=1}^n |\mu_a|^2 \prod_{b=1}^m |\nu_b|$. The integral over $y$ should be understood as spanning the whole range of complex or real values, and to include the combinatorial factors due to the invariance of $\mu_a, \nu_b$ under permutations of $y_{a'}$ or $y_{b'}$; for instance in the case of the bulk two-point function $n=2, m=0$ we have $\int d^2y \equiv \int_{\Im y_1>0} d^2y_1 + \frac{1}{2}\int_{\mathbb{R}^2} dy_1\, dy_2$. The integral over $|u|$ can be performed explicitly knowing that $\mu_a, \nu_b$ all have a factor $|u|$, the result is $\delta(i\sum_{a=1}^n(m_a+\bar{m}_a) + i\sum_{b=1}^m m_b)$. (Recall that in the $H_3^+$ model physical values of $m_a+\bar{m}_a$ and $m_b$ are pure imaginary.) The sum over $\mathrm{sgn} u$ then affects the Liouville boundary parameters, which are still given by eq. (3.19) but kept implicit in our formula. The normalization factors $N^{j_a}_{m_a,\bar{m}_a}, N^{\ell_b}_{m_b,\eta_b}$ are given in (2.12), (2.27), and we do not write the $j,\ell,m$-independent normalization factor.

A few cases are particularly simple. If $2n+m-2=0$ the $H_3^+$-Liouville relation does not involve Liouville degenerate fields. This happens for the bulk one-point function ($n=1, m=0$) and the boundary two-point function ($n=0, m=2$). If $2n+m-2=1$ the relation involves one boundary degenerate field, and therefore no singularity can occur from the collision of two degenerate fields. This happens for the bulk-boundary two-point function ($n=1, m=1$) and the boundary three-point function ($n=0, m=3$).

## 4. Boundary two-point function

The boundary two-point function for open strings living on a single $AdS_2$ brane is already known, eq. (2.28), and we reproduce it here up to irrelevant factors:

$$\left\langle \Psi^{\ell_1}(t_1|w_1)\Psi^{\ell_2}(t_2|w_2)\right\rangle_r = \delta(\ell_1+\ell_2+1)\delta(t_{12}) + \delta(\ell_1-\ell_2)\tilde{R}^H_r(\ell_1)|t_{12}|^{2\ell_1} . \tag{4.1}$$

Up to a change of the reflection number $\tilde{R}^H_r(\ell_1)$, this is actually the most general form of the two-point function which is compatible with the $\mathrm{SL}(2,\mathbb{R})$ symmetry (2.23), if the boundary fields follow the standard $\mathrm{SL}(2,\mathbb{R})$ transformation rule (2.22). And indeed, the equations in [10] which yielded that solution can also be used to derive a boundary two-point function between different branes, which is of the same form [20]. The resulting reflection number $\tilde{R}^{H\,?}_{r,r'}(\ell_1)$ however has branch cuts as a function of the boundary spin $\ell$. While this is not an inconsistency, this is certainly a strange feature.

Our relation with Liouville theory (3.18) however predicts

$$\begin{aligned}&\left\langle {}_r\Psi^{\ell_1}(t_1|w_1)_{r'}\Psi^{\ell_2}(t_2|w_2)_r\right\rangle \\ &\qquad = \delta(\ell_1+\ell_2+1)\delta(t_{12}) + \delta(\ell_1-\ell_2)\tilde{R}^H_{r,r'}(\ell_1)|t_{12}|^{2\ell_1}e^{-\frac{1}{2}(k-2)(r-r')\mathrm{sgn}t_{12}} ,\end{aligned} \tag{4.2}$$

with the $t$-basis reflection number

$$\tilde{R}^H_{r,r'}(\ell) = \frac{\pi}{\Gamma(2\ell+1)}\frac{R^L_{\frac{r}{2\pi b}+\frac{i}{4b},\frac{r'}{2\pi b}-\frac{i}{4b}}(\beta)}{\sin(\pi\ell - i\frac{r-r'}{2b^2})} = \frac{\pi}{\Gamma(2\ell+1)}\frac{R^L_{\frac{r}{2\pi b}-\frac{i}{4b},\frac{r'}{2\pi b}+\frac{i}{4b}}(\beta)}{\sin(\pi\ell + i\frac{r-r'}{2b^2})} , \tag{4.3}$$

with $\beta = b(\ell+1)+\frac{1}{2b}$. This reflection number is meromorphic in $\ell$, with no hint of a branch cut. And the factor $e^{-\frac{1}{2}(k-2)(r-r')\mathrm{sgn}t_{12}}$ contradicts the $\mathrm{SL}(2,\mathbb{R})$ symmetry.

We will argue that (4.2) is actually the correct $H_3^+$ boundary two-point function, and that the result of [20] is incorrect because it relies on erroneous symmetry assumptions. We will indeed show that the $H_3^+$ boundary condition changing operators should not belong to representations of $\mathrm{SL}(2,\mathbb{R})$ but rather to representations of the universal covering group $\widetilde{SL}(2,\mathbb{R})$.

NB: In this section we omit the dependence of two-point function in the worldsheet coordinates $w_1, w_2$. This dependence is always a factor $|w_1-w_2|^{-2\Delta_{\ell_1}}$.

### 4.1 $\widetilde{SL}(2,\mathbb{R})$ symmetry

Let us investigate how the assumption of $\widetilde{SL}(2,\mathbb{R})$ symmetry would constrain the boundary two-point function. To begin with, we study the possible actions of that group on the boundary fields ${}_r\Psi^\ell(t|w)_{r'}$.

Consider a timelike coordinate $T$ on $\widetilde{SL}(2,\mathbb{R})$ such that $T(\mathrm{id}) = 0$ and $T(-\mathrm{id}) = 1$. (As a manifold, $\widetilde{SL}(2,\mathbb{R})$ is identical to the Anti-de Sitter space $AdS_3$.) Then the set of $\widetilde{SL}(2,\mathbb{R})$ elements such that $0 \leq T < 1$ can be identified with the group $\mathrm{SL}(2,\mathbb{R})/\{\mathrm{id},-\mathrm{id}\}$. We parametrize elements of $\widetilde{SL}(2,\mathbb{R})$ as $G = (g,[T])$ where $g$ is an element of the group $\mathrm{SL}(2,\mathbb{R})/\{\mathrm{id},-\mathrm{id}\}$, and $[T]$ is the integer part of $T$.

The natural action of the group $\widetilde{SL}(2,\mathbb{R})$ on the parameter $t$ is simply $(g,[T])\cdot t = g\cdot t$. It is however possible to define an action of $\widetilde{SL}(2,\mathbb{R})$ on the $t$-basis fields $\Psi^\ell(t)$ which does not reduce to the ordinary $\mathrm{SL}(2,\mathbb{R})$ action $(g,[T])\cdot\Psi^\ell(t) = |ct-d|^{2\ell}\Psi^\ell(g\cdot t)$ as follows: for an $\widetilde{SL}(2,\mathbb{R})$ group element $G=(g,[T])$ and a real number $t$ consider the number $N$ of times $g\cdot t$ crosses $t=+\infty$ when $G$ continuously varies from $G=\mathrm{id}_{\widetilde{SL}(2,\mathbb{R})}=(\mathrm{id},0)$ to $G=(g,[T])$. Then for any fixed number $\kappa$ the following is an action of $\widetilde{SL}(2,\mathbb{R})$ on $t$-basis fields:

$$G\cdot\Psi^\ell(t) = (g,[T])\cdot\Psi^\ell(t) = |ct-d|^{2\ell}e^{\kappa N(g,[T],t)}\Psi^\ell(g\cdot t)\ . \tag{4.4}$$

How would invariance under such $\widetilde{SL}(2,\mathbb{R})$ transformations constrain the boundary two-point function? Using $N(g,[T],t)=[T]+\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t-d/c)$, we have

$$\begin{aligned}
&\left\langle G\cdot{}_r\Psi^{\ell_1}(t_1|w_1)_{r'}\ G\cdot{}_{r'}\Psi^{\ell_2}(t_2|w_2)_r\right\rangle\\
&= |ct_1-d|^{2\ell_1}|ct_2-d|^{2\ell_2}e^{\frac{1}{2}\kappa(\mathrm{sgn}[t_1-d/c]-\mathrm{sgn}[t_2-d/c])}\left\langle{}_r\Psi^{\ell_1}(g\cdot t_1|w_1)_{r'}\Psi^{\ell_2}(g\cdot t_2|w_2)_r\right\rangle\\
&= |ct_1-d|^{2\ell_1}|ct_2-d|^{2\ell_2}e^{\frac{1}{2}\kappa(\mathrm{sgn}t_{12}-\mathrm{sgn}[g\cdot t_1-g\cdot t_2])}\left\langle{}_r\Psi^{\ell_1}(g\cdot t_1|w_1)_{r'}\Psi^{\ell_2}(g\cdot t_2|w_2)_r\right\rangle .
\end{aligned}\tag{4.5}$$

The requirement that this equals $\left\langle{}_r\Psi^{\ell_1}(t_1|w_1)_{r'}\Psi^{\ell_2}(t_2|w_2)_r\right\rangle$ leads to

$$\left\langle{}_r\Psi^{\ell_1}(t_1|w_1)_{r'}\Psi^{\ell_2}(t_2|w_2)_r\right\rangle = \delta(\ell_1+\ell_2+1)\delta(t_{12})+\delta(\ell_1-\ell_2)\tilde{R}^H_{r,r'}(\ell_1)|t_{12}|^{2\ell_1}e^{-\frac{1}{2}\kappa\mathrm{sgn}t_{12}}, \tag{4.6}$$

for some $t$-basis reflection number $\tilde{R}^H_{r,r'}(\ell_1)$. Therefore, the two-point function (4.2) derived from the $H_3^+$-Liouville relation is compatible with $\widetilde{SL}(2,\mathbb{R})$ symmetry provided the boundary fields transform as eq. (4.4) with

$$\kappa = (k-2)(r-r')\ . \tag{4.7}$$

We have thus found a nice geometrical interpretation for the two-point function derived from the $H_3^+$-Liouville relation. This is of course not in itself evidence for the correctness of that relation. We will look for such evidence in the comparison with N=2 Liouville theory, and in the classical analysis of the $H_3^+$ sigma model.

### 4.2 Comparison with N=2 Liouville theory

An $H_3^+$ mod U(1) coset model can be obtained from the $H_3^+$ model by gauging, and this coset model is known to be identical to the 2d black hole coset model $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ [2]. It is also known that the N=2 supersymmetric version of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset is related via mirror symmetry to $N=2$ Liouville theory [21–23]. The boundary two-point function on maximally symmetric D-branes in N=2 Liouville theory with central charge $c=3+\frac{6}{k-2}$ is thus expected to be related to the boundary two-point function on our $AdS_2$ branes in the $H_3^+$ model at level $k$. We will not try to check this expectation in full detail, rather we will focus on the non-trivial part of the expected relation, namely the relation between the boundary reflection coefficients in the $H_3^+$ model and N=2 Liouville theory.

The boundary reflection coefficient in N=2 Liouville theory was determined in [24]. The D-branes which should be compared to the $AdS_2$ branes in $H_3^+$ are the B-branes [15]. The relation between the parameters $r$ of our $AdS_2$ branes and the parameters $J$ of the N=2 Liouville B-branes can be deduced from the explicit formulas for the corresponding one-point functions: $r = -\frac{i\pi}{k-2}(2J+1)$. The boundary fields which span the spectrum of open strings between such B-branes are called $B_m^{\ell(s)}, \lambda B_m^{\ell(s)}, \bar{\lambda} B_m^{\ell(s)}, \lambda\bar{\lambda} B_m^{\ell(s)}$, where $\ell$ and $m$ correspond to the $H_3^+$ boundary spin and $m$-basis momentum, $s$ is a fermionic label which we will ignore because the $s$-dependence of the N=2 Liouville boundary two-point function is trivial, and $\lambda, \bar{\lambda}$ are boundary fermions such that $\lambda\bar{\lambda} + \bar{\lambda}\lambda = 1$. We will compare the spectrum of open strings in $H_3^+$ with the bosonic sector of the N=2 Liouville boundary spectrum; for each choice of $\ell, m$ this sector is two-dimensional and spanned by $\bar{\lambda}\lambda B_m^{\ell(s)}, \lambda\bar{\lambda} B_m^{\ell(s)}$.

Let us write explicitly the reflection matrix for such N=2 Liouville boundary fields [24] (section 6.2 therein) with our notations and our own field normalizations chosen for later convenience. (Changing field normalizations amounts to conjugating the matrix $\mathcal{M}$ with a diagonal matrix.)

$$\begin{pmatrix} \lambda\bar{\lambda} B_m^\ell \\ \bar{\lambda}\lambda B_m^\ell \end{pmatrix} = \frac{4}{\pi}\Gamma(-\ell+m)\Gamma(-\ell-m)\Gamma(2\ell+1)\tilde{R}^H_{r,r'}(\ell) \times \mathcal{M} \begin{pmatrix} \lambda\bar{\lambda} B_m^{-\ell-1} \\ \bar{\lambda}\lambda B_m^{-\ell-1} \end{pmatrix}, \quad (4.8)$$

$$\mathcal{M} = \begin{pmatrix} \sum_\pm \pm e^{\mp\frac{r-r'}{2b^2}} \sin\pi(m\pm\ell) & e^{-i\pi m} e^{\frac{r-r'}{2b^2}} \sin 2\pi\ell \\ e^{i\pi m} e^{-\frac{r-r'}{2b^2}} \sin 2\pi\ell & \sum_\pm \mp e^{\pm\frac{r-r'}{2b^2}} \sin\pi(m\pm\ell) \end{pmatrix}.$$

The $m$-basis boundary fields $\Psi^\ell_{m,\eta}$ of the $H_3^+$ model were defined in (2.26). Our $H_3^+$ boundary two-point function (4.2) has the following form in the $m$-basis:

$$\begin{aligned} \left\langle {}_r\Psi^{\ell_1}_{m_1,\eta_1}(w_1)_{r'}\Psi^{\ell_2}_{m_2,\eta_2}(w_2)_r \right\rangle &= \delta(i(m_1+m_2)) \times \Big[\delta(\ell_1+\ell_2+1)2\pi\delta_{\eta_1\eta_2} \\ &+ \delta(\ell_1-\ell_2)\frac{4}{\pi}\Gamma(-\ell_1+m_1)\Gamma(-\ell_1-m_1)\cos\tfrac{\pi}{2}(\ell_1-m_1+\eta_1)\cos\tfrac{\pi}{2}(\ell_1+m_1+\eta_2) \\ &\times i^{\eta_1+\eta_2}\Gamma(2\ell_1+1)\tilde{R}^H_{r,r'}(\ell_1)\left\{(-1)^{\eta_1}\sin(\pi\ell+i\tfrac{r-r'}{2b^2}) + (-1)^{\eta_2}\sin(\pi\ell-i\tfrac{r-r'}{2b^2})\right\}\Big]. \end{aligned} \quad (4.9)$$

If we now assume the following identification between the N=2 Liouville fields $\lambda\bar{\lambda}B_m^\ell, \bar{\lambda}\lambda B_m^\ell$ and the $H_3^+$ model fields $\Psi^\ell_{m,\eta}$, which involves an implicit Wick rotation of the allowed values of $m$,

$$\lambda\bar{\lambda}B_m^\ell \simeq \Psi^\ell_{m,0} + \Psi^\ell_{m,1} = 2\int_0^\infty dt\ t^{-\ell-1+m}\Psi^\ell(t), \quad (4.10)$$

$$\bar{\lambda}\lambda B_m^\ell \simeq e^{i\pi m}(\Psi^\ell_{m,0} - \Psi^\ell_{m,1}) = 2e^{i\pi m}\int_{-\infty}^0 dt\ |t|^{-\ell-1+m}\Psi^\ell(t), \quad (4.11)$$

then the $H_3^+$ reflection matrix deduced from our $m$-basis boundary two-point function (4.9) agrees with the N=2 Liouville boundary reflection matrix (4.8).



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### 4.3 Classical analysis

We should be able to study such a basic property of the theory of open strings in $H_3^+$ as its symmetry group without solving the full quantum theory. In the cases of closed strings and open strings which preserve boundary conditions, the minisuperspace limit reduces our conformal field theory to the quantum mechanics of a point particle in $H_3^+$ and $AdS_2$ respectively, and therefore gives substantial insight into the spectrum and symmetry properties. However, the theory of open strings stretched between two different $AdS_2$ branes does not have such a minisuperspace limit, because such open strings can not shrink to point particles. However, we will be able to gain some insight from analyzing their classical worldsheet dynamics.

In order to predict the symmetry group, we should derive the spectrum of a timelike generator $R$ of the Lie algebra $s\ell_2(\mathbb{R})$. (Such a generator geometrically acts as a rotation of the $AdS_2$ branes.) Indeed, such a generator must satisfy $\exp 2\pi i R = -\mathrm{id}$ if the symmetry group is $\mathrm{SL}(2,\mathbb{R})$. On the other hand, no such relation exists in the universal covering group $\widetilde{SL}(2,\mathbb{R})$. Nevertheless, the transformation law (4.4) of the boundary fields suggests that the value of $\exp 2\pi i R$ applied to such fields should be $\exp 2\pi i R = e^{(k-2)(r-r')}$. The operator $\exp 2\pi i R$ is indeed identified with the $\widetilde{SL}(2,\mathbb{R})$ group element $G = (\mathrm{id}, 1)$, and for any real number $t$ we have $N(\mathrm{id}, 1, t) = 1$. The spectrum of the quantum operator $R$ is therefore expected to be

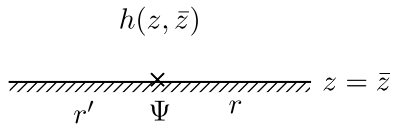
$$Spec(R) = (k-2)\frac{r-r'}{2\pi i} + \mathbb{Z} \ . \tag{4.12}$$

Of course, we do not expect the classical analysis to fully reproduce this spectrum, and in particular not the $\mathbb{Z}$ quantization. In order to show that the symmetry group is $\widetilde{SL}(2,\mathbb{R})$ and not $\mathrm{SL}(2,\mathbb{R})$, it is enough to demonstrate that the spectrum is not purely real. We will actually even find indications of an imaginary part proportional to $r - r'$.

In principle one can obtain the full set of classical solutions of the $H_3^+$ sigma-model, but it is not easy to extract predictions for the spectrum of the rotation generator $R$. This is due to the pure imaginary $B$-field in the theory on worldsheets with Lorentzian signature which prevents classical strings from evolving normally in time. On the other hand, the model on Euclidean worldsheets has many classical solutions, but it is not obvious how to relate the spectrum of $R$ evaluated on classical solutions with the quantum spectrum (4.12). We will avoid these subtleties by considering a classical solution which does not depend on the worldsheet time and therefore makes sense for both signatures. Up to simple symmetry transformations, this is actually the unique time-independent solution:

$$h = \exp \Omega \left( r + (r' - r)\frac{\sigma}{\pi} \right) \ , \tag{4.13}$$

where $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma$ is the space-like coordinate on the worldsheet. The complex coordinate on the upper half-plane worldsheet is $z = e^{\tau + i\sigma}$; our solution corresponds to inserting a boundary operator at $z = 0$:




Our solution is easily found to satisfy the following requirements:

1. Solving the bulk equations of motion. This is because $h$ can be factorized into holomorphic and antiholomorphic factors.

2. Solving the boundary conditions at $z = \bar{z}$. In terms of the currents

$$J = k\partial h h^{-1} \quad , \quad \bar{J} = k h^{-1}\bar{\partial} h = J^\dagger \, , \tag{4.14}$$

these boundary conditions are of the type

$$J^\dagger \Omega^\dagger + \Omega J \underset{z=\bar{z}}{=} 0 \, . \tag{4.15}$$

This implies the vanishing of the derivative of $\mathrm{Tr}\ \Omega h$ along the boundary, so that

$$\mathrm{Tr}\ \Omega h \underset{z=\bar{z}}{=} \begin{cases} 2\sinh r \ \ , \ \Re z > 0 \\ 2\sinh r' \ , \ \Re z < 0 \end{cases} \, , \tag{4.16}$$

as required by the definition of the brane parameters $r, r'$ (2.18).

3. Corresponding to an affine primary field insertion at $z = 0$. This means that the currents behave as

$$J(z) = \frac{k}{z} j_0 + k \sum_{n=1}^{\infty} j_{-n} z^{n-1} \, . \tag{4.17}$$

We can now evaluate the values of the conserved momenta associated to $s\ell(\mathbb{R})$ transformations:

$$i \int_0^\pi e^\tau d\sigma \ \left( \Omega^{-1} J^\dagger \Omega + J \right) = k j_0 = k \frac{r - r'}{2\pi i} \Omega \, . \tag{4.18}$$

The matrix $\frac{1}{2}\Omega$ satisfies $\exp 2\pi i (\frac{1}{2}\Omega) = -\mathrm{id}$ and can therefore be identified with the $R$ generator of the compact, timelike direction of $s\ell(\mathbb{R})$. The associated conserved charge of our classical solution is

$$R^{cl} = k \frac{r - r'}{2\pi i} \, . \tag{4.19}$$

This agrees with the imaginary part of the spectrum of the quantum operator $R$ (4.12), up a term which is subleading as $k \to \infty$. This is consistent with the classical analysis becoming reliable only in the large $k$ limit, since $k$ appears as a factor in the $H_3^+$ action (2.4) and therefore plays the rôle of the inverse Planck constant.

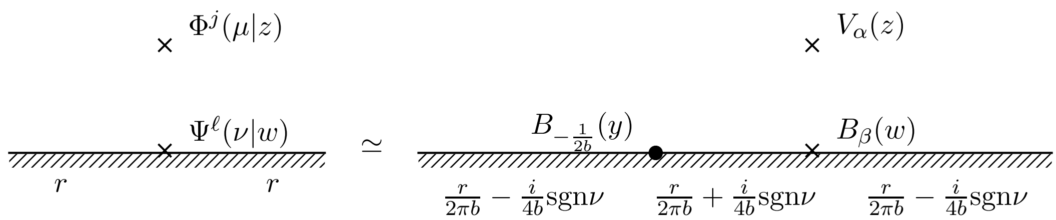
## 5. Bulk-boundary two-point function

Like the boundary two-point function, the case of the bulk-boundary two-point function will provide a nontrivial check of our expression for $H_3^+$ disc correlators in terms of Liouville theory. We will indeed use a minisuperspace analysis to independently predict the large level limit of the bulk-boundary two-point function.

According to the formula (3.18), the $H_3^+$ bulk-boundary two-point function is

$$\left\langle \Phi^j(\mu|z)\Psi^\ell(\nu|w)\right\rangle_r \propto \delta(\mu+\bar{\mu}+\nu)|u| \left|\frac{(z-\bar{z})(z-w)(\bar{z}-w)}{(w-y)(z-y)(\bar{z}-y)}\right|^{\frac{k-2}{2}} \\ \times \left\langle V_\alpha(z)_{\frac{r}{2\pi b}+\frac{i}{4b}\mathrm{sgn}\nu} B_\beta(w)_{\frac{r}{2\pi b}-\frac{i}{4b}\mathrm{sgn}\nu} B_{-\frac{1}{2b}}(y)\right\rangle , \tag{5.1}$$

where the Liouville momenta $\alpha, \beta$ are functions of $j, \ell$ (3.5), the position $y = -\frac{\mu\bar{z}w+\bar{\mu}zw+\nu z\bar{z}}{\mu z+\bar{\mu}\bar{z}+\nu w}$ of the Liouville degenerate field is the zero of the function $\varphi(t)$ (3.6), we use $u = \mu z + \bar{\mu}\bar{z} + \nu w$ (3.7), and we omit the numerical factors. Here is a picture of this $H_3^+$-Liouville relation:




The Liouville boundary parameter is therefore controlled by $\nu$, which we spell out explicitly in terms of the separated variables $(u, y)$ thanks to eq. (3.10):

$$\nu = u\frac{w-y}{|w-z|^2} . \tag{5.2}$$

### 5.1 $\mathrm{SL}(2,\mathbb{R})$ symmetry

We first check that our formula for the bulk-boundary two-point function obeys the $\mathrm{SL}(2,\mathbb{R})$ symmetry requirement (2.23). The general solution to this requirement is

$$\left\langle \Phi^j(x|z)\Psi^\ell(t|w)\right\rangle_r = |z-w|^{-2\Delta_\ell}|z-\bar{z}|^{-2\Delta_j+\Delta_\ell} \\ \times |x+it|^{2\ell}|x+\bar{x}|^{2j-\ell}\frac{\Gamma(-2j-\ell-1)}{2\pi}\sum_\pm B^H_{r,\pm}(j,\ell)\ e^{\pm i\frac{\pi}{2}(2j+\ell+1)\mathrm{sgn}\Re x}. \tag{5.3}$$

Like in the case of the bulk one-point function (which is obtained for $\ell = 0$), the $\mathrm{SL}(2,\mathbb{R})$ symmetry allows an arbitrary dependence on $\mathrm{sgn}\Re x$. Here we choose $e^{\pm i\frac{\pi}{2}(2j+\ell+1)\mathrm{sgn}\Re x}$ as a basis of functions of $\mathrm{sgn}\Re x$, and we introduce the *two $H_3^+$ bulk-boundary structure constants* $B^H_{r,\pm}(j,\ell)$. The factor $\frac{\Gamma(-2j-\ell-1)}{2\pi}$ is chosen for later convenience.

We now transform this bulk-boundary two-point function into the $\mu$-basis (defined by equations (2.10), (2.24)) for the purpose of the comparison with the formula predicted by our $H_3^+$-Liouville relation. The Fourier integral over $(x, t)$ can be performed by making the change of variables $x = x' - it$ and then parametrizing $x' \in \mathbb{C}$ in terms of real variables $\sigma, \tau$ such that $x' = \sigma(i\tau - \bar{\mu})$. (Then the integral over $\sigma$ is of the type (B.3).)

$$\begin{aligned}\left\langle \Phi^j(\mu|z)\Psi^\ell(\nu|w)\right\rangle_r &= |z-\bar{z}|^{\Delta_\ell - 2\Delta_j}|z-w|^{-2\Delta_\ell} \\ &\times \delta(\mu+\bar{\mu}+\nu)\ |\mu|^{2j+2}|\nu|^{-\ell}\int_{-\infty}^{\infty} d\tau\ |\tau|^{-2j-\ell-2}|\tau - i\mu|^{2\ell} B^H_{r,\mathrm{sgn}\tau}(j,\ell)\ . \end{aligned} \tag{5.4}$$

The remaining integral over $\tau$ converges provided $\mu$ is not pure imaginary. It can be performed using the integral formula (B.1) which yields:

$$\begin{aligned}\left\langle \Phi^j(\mu|z)\Psi^\ell(\nu|w)\right\rangle_r &= |z-\bar{z}|^{\Delta_\ell - 2\Delta_j}|z-w|^{-2\Delta_\ell} \\ &\times \delta(\mu+\bar{\mu}+\nu)\ |\mu|^{\ell+1}|\nu|^{-\ell}\frac{\Gamma(-2j-1-\ell)\Gamma(2j+1-\ell)}{\Gamma(-2\ell)}\sum_{\pm} B^H_{r,\pm}(j,\ell)F^{\pm}_{j,\ell}(\mu)\,, \end{aligned} \tag{5.5}$$

where we define

$$F^{\pm}_{j,\ell}(\mu) \equiv \left(\frac{1}{2} \pm \frac{1}{2}\frac{\Im\mu}{|\mu|}\right)^{\ell+\frac{1}{2}} F\left(2j+\frac{3}{2}, -2j-\frac{1}{2}, \frac{1}{2}-\ell; \frac{1}{2}\mp\frac{1}{2}\frac{\Im\mu}{|\mu|}\right)\ . \tag{5.6}$$

The Liouville correlator in (5.1) can be decomposed into Liouville structure constants, and conformal blocks which capture all the dependence on the worldsheet coordinates $z, w, y$. The properties of the relevant blocks have been studied in [25], and they are proportional to the functions $F^{\pm}_{j,\ell}(\mu)$ in (5.6).[5] What is however not obvious, but necessary for the $\mathrm{SL}(2,\mathbb{R})$ symmetry, is that the coefficients of this decomposition are completely independent of $\mu, \nu$, in spite of the $\mathrm{sgn}\nu$ dependence of the Liouville boundary parameter.

In order to write the decomposition explicitly, let us consider the Liouville factorization limit $y \to w$ when the degenerate boundary field $B_{-\frac{1}{2b}}$ collides with $B_\beta$. This limit corresponds to $\nu = 0$, and therefore $\mu$ pure imaginary (using $\mu+\bar{\mu}+\nu = 0$). The behaviour of our conformal blocks $F^{\pm}_{j,\ell}(\mu)$ in this limit is actually determined by $\lim_{\nu\to 0} \mathrm{sgn}\Im\mu = -\mathrm{sgn}u$. Namely, $F^+$ is regular if $\mathrm{sgn}u = -$ and $F^-$ is regular if $\mathrm{sgn}u = +$. The Liouville correlator in (5.1) is then decomposed into regular blocks, and structure constants where the boundary parameters can be determined from the identity $\mathrm{sgn}\nu = \mathrm{sgn}u\ \mathrm{sgn}(w-y)$. We thus find the following decomposition, where $\epsilon = \mathrm{sgn}u$:

$$\begin{aligned}\left\langle \Phi^j(\mu|z)\Psi^\ell(\nu|w)\right\rangle_r &= |z-\bar{z}|^{\Delta_\ell - 2\Delta_j}|z-w|^{-2\Delta_\ell}\delta(\mu+\bar{\mu}+\nu) \\ &\times \Bigg[ C^L_{s_\epsilon}(\beta\underset{s_{-\epsilon}}{|} -\tfrac{1}{2b}, Q-\beta-\tfrac{1}{2b}) B^L_{s_\epsilon}(\alpha, \beta+\tfrac{1}{2b})|\mu|^{\ell+1}|\nu|^{-\ell}F^{-\epsilon}_{j,\ell}(\mu) \\ &+ C^L_{s_\epsilon}(\beta\underset{s_{-\epsilon}}{|} -\tfrac{1}{2b}, Q-\beta+\tfrac{1}{2b}) B^L_{s_\epsilon}(\alpha, \beta-\tfrac{1}{2b})|\mu|^{-\ell}|\nu|^{\ell+1}F^{-\epsilon}_{j,-\ell-1}(\mu)\Bigg]\ , \end{aligned} \tag{5.7}$$

[5]According to [25], the relevant Liouville blocks are indeed powers of $|z-w|, |z-y|, |w-y|$, times hypergeometric functions of the type $F(b^{-1}(2\alpha+\beta-Q), b^{-1}(\beta-\frac{1}{2b}), 2b^{-1}(\beta-\frac{1}{2b}); \tilde{z})$ with $\tilde{z} = \frac{(z-\bar{z})(y-w)}{(z-w)(y-\bar{z})} = 1+\frac{\mu}{\bar{\mu}}$. The relation with $F^{\pm}_{j,\ell}(\mu)$ is established thanks to the quadratic transformation (B.4).

where $s_\pm = \frac{r}{2\pi b} \mp \frac{i}{4b}$, and the Liouville bulk-boundary structure constant $B^L_s(\alpha,\beta)$ (C.4) and degenerate boundary three-point structure constant $C^L_s(\beta\,\underset{s'}{|}\,-\frac{1}{2b}, Q-\beta\pm\frac{1}{2b})$ (C.7) are explicitly known.

The Liouville correlator in (5.1) is known to have an alternative decomposition [25], which leads to equation (5.7) being also valid for $\epsilon = -\mathrm{sgn}u$. (The equality of these decompositions can be exploited in order to derive a $\frac{1}{2b}$-shift relation for the Liouville bulk-boundary structure constant.) We will now use these two decompositions $\epsilon = \pm$, while rewriting the functions $F^{-\epsilon}_{j,-\ell-1}(\mu)$ in terms of $F^{\pm}_{j,\ell}(\mu)$ with the help of

$$\left|\frac{\nu}{4\mu}\right|^{2\ell+1} \frac{\Gamma(-\ell+\frac{1}{2})\Gamma(-\ell-\frac{1}{2})}{\Gamma(-\ell-2j-1)\Gamma(-\ell+2j+1)} F^{\epsilon}_{j,-\ell-1} = F^{-\epsilon}_{j,\ell} + \frac{\cos\pi 2j}{\cos\pi\ell} F^{\epsilon}_{j,\ell}\ . \tag{5.8}$$

Not forgetting $C^L_s(\beta\,\underset{s'}{|}\,-\frac{1}{2b}, Q-\beta+\frac{1}{2b}) = 1$, we find that the $H_3^+$ bulk-boundary two-point function deduced from the $H_3^+$-Liouville relation is indeed of the form (5.5) dictated by $\mathrm{SL}(2,\mathbb{R})$ symmetry, with structure constants

$$B^H_{r\pm}(j,\ell) = \frac{2^{4\ell+2}\Gamma(-2\ell)}{\Gamma(-\ell-\frac{1}{2})\Gamma(-\ell+\frac{1}{2})} B^L_{\frac{r}{2\pi b}\mp\frac{i}{4b}}(\alpha,\beta-\tfrac{1}{2b})\ . \tag{5.9}$$

(We omit numerical factors.)

### 5.2 Minisuperspace analysis

Let us compute the minisuperspace limit $k\to\infty$ of our $H_3^+$ bulk-boundary two-point function. Thanks to $\mathrm{SL}(2,\mathbb{R})$ symmetry, this reduces to computing the $k\to\infty$ limit of the structure constants $B^H_{r,\pm}(j,\ell)$ computed in [25] and reproduced in the appendix, eq. (5.9). Using the explicit formula for the Liouville bulk-boundary structure constants $B^L$ (C.4), we compute their semi-classical limit $b^2 = \frac{1}{k-2}\to 0$ (assuming $r,j,\ell$ stay fixed):

$$\begin{aligned}
\lim_{b\to 0} B^L_{\frac{r}{2\pi b}\mp\frac{i}{4b}}(\alpha,\beta-\tfrac{1}{2b}) &= 4\pi(-\mu_L\pi b^{-2}e^{2r})^{-1-j-\frac{\ell}{2}} \\
&\qquad \int_{i\mathbb{R}} \frac{dp}{2\pi i} e^{(-2r\pm i\pi)p} \frac{\Gamma(p+\ell+2j+2)\Gamma(p+\ell+1)\Gamma(-p-2j-1)\Gamma(-p)}{\Gamma(\ell+1)\Gamma(-2j-1)} \\
&= 4\pi(\mu\pi b^{-2})^{-1-j-\frac{\ell}{2}} \\
&\quad\times\Big\{ e^{\mp\frac{i\pi}{2}(2j-\ell+2)} \frac{\Gamma(-2j+\ell)\Gamma(2j+2)}{\Gamma(-2j)} e^{(2j-\ell)r} F(\ell+1,-2j+\ell,-2j;-e^{-2r}) \\
&\quad + e^{\pm\frac{i\pi}{2}(2j+\ell+2)}\Gamma(2j+\ell+2)e^{-(2j+\ell+2)r}F(\ell+1,2j+\ell+2,2j+2;-e^{-2r})\Big\}\ ,
\end{aligned} \tag{5.10}$$

where we make use of the asymptotic behaviour of the special functions (A.8)–(A.11) and of the auxiliary formula

$$e^{(-2r\pm i\pi)p} = e^{-2r}\frac{e^{\mp i\pi\ell}\sin\pi(2j+p) - e^{\mp i\pi 2j}\sin\pi(\ell+p)}{\sin\pi(2j-\ell)}\ . \tag{5.11}$$

Let us now predict the minisuperspace limit of the bulk-boundary structure constants $B^H_{r,\pm}(j,\ell)$ by an independent calculation. In the minisuperspace limit, the $H_3^+$ model path-integral reduces to the integral over worldsheet-independent elements $h$ of $H_3^+$:

$$\left\langle \Phi^j(x|z)\Psi^\ell(t|w)\right\rangle_r^{\text{mini}} \equiv \int_{H_3^+} dh\ \Phi^j(x|h)\Psi^\ell(t|h)\delta(\text{Tr}\,[h\Omega]-2\sinh r)\ . \tag{5.12}$$

Using the explicit formulas for the $H_3^+$ element $h$ (2.3) and the classical fields $\Phi^j(x|h)$ (2.5) and $\Psi^\ell(t|h)$ (2.20), the minisuperspace computation is performed as follows:

$$\begin{aligned}
&\left\langle \Phi^j(x|z)\Psi^\ell(t|w)\right\rangle_r^{\text{mini}} \\
&\quad= -\frac{2j+1}{\pi}\int d^2\gamma d\phi e^{2\phi}\delta(e^\phi(\gamma+\bar\gamma)-2\sinh r)(|x-\gamma|^2e^\phi+e^{-\phi})^{2j}(|it+\gamma|^2e^\phi+e^{-\phi})^\ell \\
&\quad= -\frac{2j+1}{2\pi}\int\frac{d^2\gamma du}{u^{2j+\ell+1}}\delta(\Re\gamma-\sinh r)(|u(x+it)-\gamma|^2+1)^{2j}(|\gamma|^2+1)^\ell \\
&\quad= |x+\bar x|^{2j-\ell}|x+it|^{2\ell} \\
&\qquad\times\frac{\Gamma(-4j-1)(2\cosh r)^{2j+\ell+1}}{\Gamma(-2j)\Gamma(-2j-1)}\int_0^\infty\frac{du}{u^{2j+\ell+2}}(u^2-2u\tanh r\ \text{sgn}\Re x+1)^{2j+\frac12},
\end{aligned} \tag{5.13}$$

where $\gamma$ was shifted $\gamma\to\gamma-it$ and rescaled $\gamma\to e^{-\phi}\gamma$, we introduced $u=e^\phi$, and we reached the last expression by the rescaling $u\to\frac{(1+|\gamma|^2)\Re x}{|x+it|^2\cosh r}u$ which allowed the integral over $\gamma$ to be performed.

The remaining integral over $u$ can be performed with the help of the formula (B.1). The minisuperspace bulk-boundary two-point function is then found to be of the form dictated by the spacetime $\text{SL}(2,\mathbb{R})$ symmetry (5.4), with the structure constants:

$$\begin{aligned}
B^{H,\text{mini}}_{r,\pm}(j,\ell) &= (2\cosh r)^{2\ell+1} \\
&\times\left\{e^{\mp\frac{i\pi}{2}(2j-\ell+2)}\frac{\Gamma(-2j+\ell)\Gamma(2j+2)}{\Gamma(-2j)}e^{(2j-\ell)r}F(\ell+1,-2j+\ell,-2j;-e^{-2r})\right. \\
&\left.+\,e^{\pm\frac{i\pi}{2}(2j+\ell+2)}\Gamma(2j+\ell+2)e^{-(2j+\ell+2)r}F(\ell+1,2j+\ell+2,2j+2;-e^{-2r})\right\}.
\end{aligned} \tag{5.14}$$

Up to a renormalization of the fields, this agrees with the prediction (5.10) from our $H_3^+$-Liouville relation.

## A. Special functions

The function $\gamma(x)$ is built from Euler's Gamma function:

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{A.1}$$

We use the special functions $\Gamma_b$, $\Upsilon_b$ and $S_b$ which usually appear in the study of Liouville theory at parameter $b>0$ and background charge $Q=b+b^{-1}$. We use the same conventions as [26], where some more details can be found. The following definitions are valid for $0 < \Re x < Q$:

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q/2-x)^2}{2}e^{-t} - \frac{Q/2-x}{t}\right], \quad \text{(A.2)}$$

$$\log\Upsilon_b = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right], \quad \text{(A.3)}$$

$$\log S_b = \int_0^\infty \frac{dt}{t}\left[\frac{\sinh(\frac{Q}{2}-x)t}{2\sinh(\frac{bt}{2})\sinh(\frac{t}{2b})} - \frac{(Q-2x)}{t}\right]. \quad \text{(A.4)}$$

These functions can be extended to a meromorphic function on the complex plane thanks to the shift equations

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi}b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x), \qquad \Gamma_b(x+1/b) = \frac{\sqrt{2\pi}b^{-\frac{x}{b}+\frac{1}{2}}}{\Gamma(x/b)}\Gamma_b(x) \quad \text{(A.5)}$$

$$\Upsilon_b(x+b) = \frac{\Gamma(bx)}{\Gamma(1-bx)}b^{1-2bx}\Upsilon_b(x), \qquad \Upsilon_b(x+1/b) = \frac{\Gamma(x/b)}{\Gamma(1-x/b)}b^{2x/b-1}\Upsilon_b(x) \quad \text{(A.6)}$$

$$S_b(x+b) = 2\sin(\pi bx)S_b(x), \qquad S_b(x+1/b) = 2\sin(\pi x/b)S_b(x) \quad \text{(A.7)}$$

The three special functions are related: $S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$ and $\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}$.

Using the integral representations for the special functions, one can study their behaviour for $b \to 0$ while keeping the quantity $x$ fixed:

$$\Gamma_b(bx) \to (2\pi b^3)^{\frac{1}{2}(x-\frac{1}{2})}\Gamma(x) \quad , \quad \Gamma_b(Q-bx) \to \left(\frac{b}{2\pi}\right)^{\frac{1}{2}(x-\frac{1}{2})}, \quad \text{(A.8)}$$

$$\Upsilon_b(bx) \to \frac{1}{b^{x-\frac{1}{2}}\Gamma(x)}, \quad \text{(A.9)}$$

$$S_b(bx) \to (2\pi b^2)^{x-\frac{1}{2}}\Gamma(x) \quad , \quad S_b(\frac{1}{2b}+bx) \to 2^{x-\frac{1}{2}}, \quad \text{(A.10)}$$

$$\prod_\pm S_b(\frac{1}{2b}+bx \pm i\frac{r}{\pi b}) \to \left(\frac{\cosh r}{\sqrt{2}}\right)^{1-2x}. \quad \text{(A.11)}$$

## B. Useful formulas

The following formula [27] is used in section 5.

$$\int_0^\infty dx x^\alpha (1 \pm 2x\tanh r + x^2)^\beta = \frac{\Gamma(\alpha+1)\Gamma(-2\beta-\alpha-1)}{\Gamma(-2\beta)}$$
$$\times (2\cosh r)^{-\beta-\frac{1}{2}}e^{\pm r(\beta+\frac{1}{2})}F\left(-\alpha-\beta-\frac{1}{2}, \alpha+\beta+\frac{3}{2}, \frac{1}{2}-\beta; \frac{1}{e^{\pm 2r}+1}\right). \quad \text{(B.1)}$$

The following formulas are useful for transforming some correlators in the $(x,t)$ basis into correlators in the $(\mu,\nu)$ basis.

$$\int_{\mathbb{C}} d^2x\, e^{i\Im\mu x}|x|^{2\alpha} = \pi\gamma(\alpha+1)|\mu|^{-2\alpha-2}, \quad \text{(B.2)}$$

$$\int_{\mathbb{R}} dt\, f(\mathrm{sgn}t)|t|^\alpha e^{-it\nu} = |\nu|^{-\alpha-1}\Gamma(\alpha+1)\left[f(\mathrm{sgn}\nu)e^{-i\frac{\pi}{2}(\alpha+1)} + f(-\mathrm{sgn}\nu)e^{i\frac{\pi}{2}(\alpha+1)}\right]. \quad \text{(B.3)}$$

The conformal blocks which are relevant for section 5 involve hypergeometric functions which can undergo a quadratic transformation:

$$F(a,b,2b;z) \;=\; \left(\tfrac{1}{2}+\tfrac{1}{4}\tfrac{2-z}{\sqrt{1-z}}\right)^{\frac{1}{2}-b}(1-z)^{-\frac{a}{2}} \times F\left(b-a+\tfrac{1}{2},a-b+\tfrac{1}{2},b+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{4}\tfrac{2-z}{\sqrt{1-z}}\right)\,. \tag{B.4}$$

## C. Some Liouville theory formulas

We mostly follow conventions of [26]. We consider Liouville theory with parameter $b>0$, background charge $Q=b+b^{-1}$, central charge $c=1+6Q^2$, and interaction strength $\mu_L$.

One-point function on a disc:

$$\begin{aligned}\left\langle V_\alpha(z)\right\rangle_s \;=&\; |z-\bar{z}|^{-2\Delta_\alpha}(\pi\mu_L\gamma(b^2))^{\frac{Q-2\alpha}{2b}} \\ &\times\frac{\Gamma(1-b(Q-2\alpha))\Gamma(1-b^{-1}(Q-2\alpha))}{-\pi 2^{\frac{1}{4}}(2\alpha-Q)}\cosh 2\pi s(Q-2\alpha)\;.\end{aligned} \tag{C.1}$$

Boundary reflection coefficient and two-point function:

$$\left\langle {}_sB_{\beta_1}(w_1)_{s'}B_{\beta_2}(w_2)_s\right\rangle \;=\; |w-\bar{w}|^{-2\Delta_{\beta_1}}\left[\delta(\beta_1+\beta_2-Q)+R^L_{s,s'}(\beta_1)\delta(\beta_1-\beta_2)\right]\,, \tag{C.2}$$

$$R^L_{s,s'}(\beta) \;=\; \left[\pi\mu_L\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-2\beta}{2b}}\frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}\prod_{\pm\pm}S_b\left(Q-\beta+i(\pm s\pm s')\right)\;.$$

Bulk-boundary two-point function [25]:

$$\left\langle V_\alpha(z)B_\beta(w)\right\rangle_s \;=\; |z-\bar{z}|^{\Delta_\beta-2\Delta_\alpha}|z-w|^{-2\Delta_\beta}B^L_s(\alpha,\beta)\,, \tag{C.3}$$

$$\begin{aligned}B^L_s(\alpha,\beta) \;=&\; i2^{-\frac{1}{4}}\left[\pi\mu_L\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-2\alpha-\beta}{2b}}\frac{\Gamma_b^3(Q-\beta)\Gamma_b(2Q-2\alpha-\beta)\Gamma_b(2\alpha-\beta)}{\Gamma_b(Q)\Gamma_b(\beta)\Gamma_b(Q-2\beta)\Gamma_b(2\alpha)\Gamma_b(Q-2\alpha)} \\ &\times\int_{-i\infty}^{i\infty}dp\; e^{-4\pi sp}\prod_\pm\frac{S_b(\alpha+\frac{\beta-Q}{2}\pm p)}{S_b(\alpha-\frac{\beta-Q}{2}\pm p)}\,.\end{aligned}$$

From this, one can deduce the bulk-boundary OPE of a bulk degenerate field $V_{-\frac{1}{2b}}$ which is relevant for our continuity assumption (3.17). There is a subtlety: due to the pole structure of $B^L_s(\alpha,\beta)$, the degenerate limit of the bulk-boundary OPE yields

$$B^L_s(-\tfrac{1}{2b},Q) = \lim_{\alpha\to-\frac{1}{2b}}\;\operatorname*{Res}_{\beta=b-2\alpha}\;B^L_s(\alpha,\beta)\,, \tag{C.4}$$

instead of the incorrect formula $B^L_s(-\frac{1}{2b},Q)\stackrel{?}{=}\lim_{\alpha\to-\frac{1}{2b}}\operatorname*{Res}_{\beta=Q}B^L_s(\alpha,\beta)$ which one might naively have expected. The correct result is

$$B^L_s(-\tfrac{1}{2b},Q) = 2b^{-2}\left[\pi\mu_L\gamma(b^2)\right]^{\frac{1}{2b^2}}\frac{\Gamma(-1-2b^{-2})}{\Gamma(-b^{-2})}\cosh 2\pi b^{-1}s\;. \tag{C.5}$$

Operator product expansion of a degenerate boundary field:

$$
{}_{s_+}B_\beta(w)_{s_-}B_{-\frac{1}{2b}}(y)_{s_+} \sim |w-y|^{b^{-1}(Q-\beta)} C^L_{s_+}(\beta \underset{s_-}{|} -\tfrac{1}{2b}, Q-\beta-\tfrac{1}{2b})_{s_+}B_{\beta+\frac{1}{2b}}(w)_{s_+}
$$
$$
+ |w-y|^{b^{-1}\beta} C^L_{s_+}(\beta \underset{s_-}{|} -\tfrac{1}{2b}, Q-\beta+\tfrac{1}{2b})_{s_+}B_{\beta-\frac{1}{2b}}(w)_{s_+}\,, \quad \text{(C.6)}
$$

with the degenerate boundary structure constants

$$
C^L_{s_+}(\beta \underset{s_-}{|} -\tfrac{1}{2b}, Q-\beta+\tfrac{1}{2b}) = 1
$$
$$
C^L_{s_+}(\beta \underset{s_-}{|} -\tfrac{1}{2b}, Q-\beta-\tfrac{1}{2b}) = R^L_{s_-,s_+}(\beta)R^L_{s_+,s_+}(Q-\beta-\tfrac{1}{2b}) = \frac{2b^{-2}}{\pi}\left[\pi\mu_L\gamma(b^{-1})\right]^{\frac{1}{2b^2}} \times
$$
$$
\times\Gamma(1-2b^{-1}\beta)\Gamma(2b^{-1}\beta-b^{-1}Q)\cos\pi(b^{-1}\beta-\tfrac{b^{-1}Q}{2})
$$
$$
\times\cos\pi(b^{-1}\beta-\tfrac{b^{-1}Q}{2}\mp 2ib^{-1}s_+)\,,
$$

where $s_+ = s_- \pm \frac{i}{2b}$. The particular case of the OPE of two degenerate boundary fields $\beta = -\frac{1}{2b}$ is relevant for our continuity assumption (3.16).

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